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A SEMI-GRAPHICAL METHOD OF APPROXIMATING AMPLITUDE FUNCTIONS OF FREQUENCY IN CONTINUED FRACTION FORM

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Elliott Carr Cutler Jr.

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A SEMI-GRAPHICAL METHOD

OF APPROXIMATING AMPLITUDE FUNCTIONS OF FREQUENCY

IN CONTINUED FRACTION FORM

Approved:

B. J. Dasher

Kendall L. Su

Daniel C. Fielder

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SUMMARY

In this study, a method is advanced for approximating amplitude functions of frequency in continued fraction form. The prescribed function to be approximated is given in graphical or tabulated form, and no analytic express for it is known. Tolerance limits for a successful approximation are also prescribed. The prescribed function may represent either the squared magnitude of a transfer function or the real part of a driving-point impedance. The desired approximating function is to be a rational function in the variable ω^2, non-negative for all real values of ω. The presumed existence of an acceptable rational fraction approximation imposes certain restrictions on the behavior of the given function and tolerance limits.

The method consists of expanding the prescribed function in a continued fraction, the elements of the continued fraction being rational functions of ω^2. A series of approximants to the continued fraction is obtained, each approximant being the function obtained by discarding all terms after a certain element in the expansion. By their construction the approximants are rational functions of ω^2. The expansion process is continued until an approximant is generated that satisfactorily approximates the prescribed function.

Since the prescribed function is a tabulated one, the calculations at each stage of the procedure must be performed for a selection of representative frequencies, usually chosen at intervals of one-tenth of

a decade throughout the frequency range of primary interest. The removal of the initial element from the prescribed function therefore results in a remainder which is also a tabulated function. The removal of the next element from this remainder leaves a second tabulated remainder, and so forth. Each remainder is plotted graphically as an aid in selecting the succeeding element. Eventually a remainder is obtained which may be discarded, leaving the desired acceptable approximant.

The purpose of the study is to investigate several policies or strategies for the selection of the elements. Particular topics treated are:

(a) the improvement of successive approximants to the prescribed function;

(b) the relation between approximants to remainders and approximants to the prescribed function;

(c) convergence of the continued fraction expansion, and the relation of its convergence limit to the value of the prescribed function;

(d) the relation between different series of approximants to the same prescribed function;

(e) the relation between the shape of element curves and the shape of corresponding remainders;

(f) the effect of element selections on the order of the final approximant;

(g) determination of the stage at which a successful approximant has been achieved; and

(h) procedures for accomplishing the calculations entirely in the logarithmic domain.

The elements used in the continued fraction expansion of the prescribed function may be general rational functions, or they may be rational functions restricted in form as, for example, monomial functions of ω^2 (which are linear when plotted to logarithmic scales). The case of general rational functions is first investigated, and it is shown that providing the products of each element and its corresponding remainder are either uniformly less than unity or uniformly greater than unity for all stages of the expansion, certain useful inequality relations result. For example, these relations may assure, providing the conditions for their existence are satisfied, that some of the following properties hold true at a particular frequency or in a band of frequencies.

(a) Certain higher-ordered approximants are better than certain lower-ordered ones.

(b) An approximant is closer to the prescribed function than the reciprocal of the last element is to the remainder it approximates.

(c) The difference between an approximant and the prescribed function is less than a value readily obtained from certain tables.

These properties are utilized in the strategy for the selection of successive elements. They may indicate at what frequencies the reciprocal of an element should be close to its remainder, and at what frequencies its value is immaterial. They may also indicate when an approximant satisfactory at all frequencies has been obtained.

The relations of the type indicated above hold at specific frequencies or in bands of frequencies where the conditions producing them are in force. Because of the tabulated nature of the prescribed function, the fact that they hold at one frequency has no bearing on whether they are valid at a neighboring one. The elements and approximants, however, are rational functions, and the values they take on at neighboring frequencies are restricted by this property. In choosing elements it is therefore necessary to consider how the prescribed function and its remainders vary with frequency; thus even though it is usually advantageous for the reciprocal of an element to be close to its corresponding remainder, it is not advantageous for it to approach this remainder too rapidly over a narrow band of frequencies.

For many reasons it is best to treat the frequency aspects of the problem with all the pertinent quantities plotted to the familiar logarithmic scales, that is, with the amplitudes of the prescribed function, elements, remainders, and approximants expressed in decibels and plotted against the abscissa, $\log \omega$. To make possible the expeditious calculation of each remainder from the preceding remainder and the appropriate element in terms of the logarithmic quantities, a calculation which involves a subtraction in the algebraic domain, the function

$$L(u) = 10 \log \left[1 - \operatorname{antilog}(u/10)\right]$$

is introduced. $L(u)$ is tabulated at intervals of 0.01 db to obviate time-consuming interpolation, and provision is made for handling the logarithms of negative quantities. The result is that all the calculations of the proposed method are either additions or subtractions, or entries into the tables. The effect on the accuracy of the method imposed by using the tables is investigated to give additional guidance in the selection of elements.

The prescribed function may be replaced by its upper and lower tolerance bounds in the calculations, and corresponding upper and lower bounds of successive remainders are then obtained. By using this capability of the method in cases where the difference between the upper and lower bounds varies in magnitude with frequency, a satisfactory approximant may be found of lower order than might otherwise be readily obtained.

In the variation of the method in which the elements are permitted to be general rational functions, the elements are chosen from a table of standard components with amplitudes given in decibels provided for this purpose, and the elements may be the sum of a selection of these components.

The study also deals with a variation of the method in which the reciprocals of the elements are restricted to have linear form in the logarithmic domain, and are chosen to be asymptotic to their corresponding remainders at infinity. The major problem presented by this variation is that of obtaining an approximant acceptable over the entire frequency spectrum when formally the method deals only with the high-frequency behavior of the various quantities. The problem is solved by starting the expansion with elements asymptotic for large ω, but subsequently shifting attention to progressively lower frequencies, at the same time adhering to certain conditions which guarantee that the high-frequency behavior of the later elements will not throw the final approximant outside the tolerance bounds. By suitable frequency transformations this method may be employed to expand the prescribed function in terms of linear elements asymptotic at frequencies other than infinity.

The principal advantages of this method of approximating graphical or tabulated prescribed functions are considered to be the following. If a particular approximant is not satisfactory:

(a) this fact may be determined without calculating the values of the approximant point by point over the entire spectrum of interest,

(b) the method indicates how to choose the succeeding element in such a way that the succeeding approximant is acceptable, or, at least, better than the preceding one, and

(c) the expansion procedure is readily continued from the point already reached, and it is not necessary to recommence the procedure from the beginning.

A disadvantage is that the order of the final approximant is not guaranteed to be the lowest possible, but in most cases it appears to be reasonable.

CHAPTER I

INTRODUCTION

The approximation problem.--One of the principal subdivisions of the field of network synthesis is the approximation problem. The subject has been explored extensively in the literature (1). The problem is to find a rational function of suitable form that approximates satisfactorily an arbitrarily prescribed function. The problem has been subdivided into classes in several ways. These include:

(a) approximation in the time domain and in the frequency domain (2),

(b) approximation of frequency characteristics, of phase characteristics, and of both simultaneously (3),

(c) approximation in the Taylor sense, in the least squares sense, in the Tchebycheff sense, and by point-coincident polynomials (4,5), and

(d) approximation for filters, for equalizers, for servomechanisms, and so forth; that is, classification by purpose, which is closely related to required accuracy and simplicity (6).

Another possible classification is a separation into graphical or semi-graphical methods, and analytical methods. Semi-graphical methods seek to obtain the desired rational function directly from the graphical of tabulated data. Analytical methods presuppose that the function to be approximated is in analytic form or has been approximated sufficiently closely by an analytic (but not rational) function.

Analytical methods have received the lion's share of attention in the literature. They are particularly pertinent for cases in which the function to be approximated is in analytic form. This occurs quite frequently, especially in cases where the related device, a filter for example, is to perform in a manner prescribed by man. It is in the nature of things for such prescriptions to be analytically expressed. In addition, analytical methods offer to engineers the attractive prospect of proceeding in a steady march from problem to solution, incorporating prescribed restraints and tolerances into the formulation enroute, and arriving at the final result without having to backtrack and cut-and-try at any stage of the procedure. Analytical methods can achieve very accurate approximations when the attendant complexity of the result is acceptable.

Semi-graphical methods are particularly applicable in the servomechanism field. Here the function to be approximated often represents the measured performance of some device, and exhibits the superior ingenuity of nature over man in devising requirements that have no obvious simple analytic expressions. At the same time the tolerance of the approximation is sufficiently relaxed by the limited accuracy of the measured data that semi-graphical methods are appropriate (7).

Semi-graphical methods.--The term "semi-graphical" is intended to indicate that the methods embraced in this classification are not purely graphical. They have in common the property of commencing from prescribed data presented in graphical or tabulated form. They are closely circumscribed by the required form and properties of the result; and analytical procedures, in addition to trial and error procedures, are available to make final improvements in the accuracy of the result.

The most common method of semi-graphical approximation was initially proposed by Baum (8). For the desired rational function he assumes the form of a ratio of polynomials, each of which is factored into a product of terms of one of the following forms:

(1) $\left[1 + \left(\frac{\omega}{\omega_k}\right)^{2n}\right]$, called Butterworth functions, or

(2) $\left[1 + \varepsilon_k T_{2n}\left(\frac{\omega}{\omega_k}\right)\right]$, in which $T_{2n}\left(\frac{\omega}{\omega_k}\right)$ is a Tchebycheff polynomial of the first kind and order 2n.

Then the logarithmic expression of the desired function, in decibels, is a sum of terms having any of the following forms:

$$(1)\ \pm 10 \log \left[1 + \left(\frac{\omega}{\omega_k}\right)^{2n}\right] \text{ or } \pm 10 \log \left[\frac{1 + \left(\frac{\omega}{\omega_k}\right)^{2n}}{1 + \left(\frac{\omega}{\omega_k}\right)^{2p}}\right]$$

$$(2)\ + 10 \log \left[1 + \varepsilon_k T_{2n}\left(\frac{\omega}{\omega_k}\right)\right] \text{ or } + 10 \log \left[\frac{1 + \varepsilon_k T_{2n}\left(\frac{\omega}{\omega_k}\right)}{1 + \varepsilon_k T_{2p}\left(\frac{\omega}{\omega_k}\right)}\right]$$

These type terms are given by Baum in graphical form plotted against the normalized logarithmic frequency scale, $\log \frac{\omega}{\omega_k}$, for various values of n, p, and ε_k. The procedure is to select from the various possible components thus made available a group whose sum approximates acceptably the prescribed function.

Linke (9) adds to the arsenal of available components functions of the form

$$+ 10 \log \left[1 + 2c \left(\frac{\omega}{\omega_k}\right)^2 + \left(\frac{\omega}{\omega_k}\right)^4\right]$$

Actually functions of this type have an equivalent representation in the Tchebycheff-type class of components advanced by Baum, but for many applications they are easier to handle in the form selected by Linke. Sets of these curves, for various values of c or a related parameter, are graphically presented in many standard texts (10). They are also applicable to the semi-graphical method proposed in this study, and a selection of them is tabulated in Appendix B.

Bresler (11) proposes functions of the form

$$10 \log \left[\frac{\sigma_o^2 + \omega^2}{\sigma_p^2 + \omega^2}\right]$$

which represent the amplitude associated with a pole-zero pair on the negative real axis. He gives graphs for calculating the contribution of such a pair to the overall response. Cutteridge (12) uses templates based on the various types of component functions mentioned above to show how the phase, delay, or step response of a circuit may be estimated from the given amplitude-frequency response. Linvill (13) shows how the accuracy of approximations obtained by the preceding methods may be improved by small shifts in the real and imaginary parts of the poles and zeros already selected, and gives an analytical procedure for obtaining the best approximation by applying the method of least squares at a selected number of critical frequencies.

The foregoing contributions comprise the most widely used method of semi-graphical approximation. In an unrelated method Saraga and Fosgate (14) describe the construction of special charts, pertinent to certain filter problems, on which key functions are linearized. The use of electrolytic tanks, employing the entirely different approach of potential analog theory, may also properly be included in the class of semi-graphical methods.

In applying the principal semi-graphical method outlined above difficulty is experienced if a number of "break" frequencies (ω_k in the expressions above) are located close together so that the overall logarithmic response at a given frequency is the sum of significant contributions from a relatively large number of components. In addition, in cases where the application of Linvill's procedure fails to obtain the necessary accuracy, the entire procedure must be repeated with a different number of poles and zeros (15). This suggests that a desirable property of an alternative method would include the characteristic that if the procedure up to a certain point had not produced a satisfactory result, it could be continued forward from that point without reevaluating the preceding steps. This characteristic suggests the behavior of successive approximants to a convergent continued fraction, and provides the point of departure for the method advanced in this study.

CHAPTER II

CONCEPT OF THE METHOD

The prescribed function.--The prescribed function which it is desired to approximate is given in graphical or tabulated form. The given data cover the frequency range of interest, which may be all, or only a significant part, of the positive frequency spectrum. The function represents the squared magnitude of a transfer function or the real part of a driving-point impedance, and the acceptable approximant which is being sought must therefore be an even function of frequency and be non-negative for all frequencies. Since data are not given for negative values of frequency, it may be assumed that the prescribed function is also an even function of frequency. Let the prescribed function be $G_o(\omega^2)$.

There are restrictions on the acceptable approximant to G_o that must be reflected in G_o. If it represents the real part of a driving-point impedance it must have no poles for real values of ω. If it represents the squared magnitude of a transfer function it can have poles of no more than double multiplicity for real ω, and those only at finite non-zero values of ω (16). Zeros must be of even multiplicity. If G_o does not conform exactly to these specifications the tolerance limits for an acceptable approximant must be sufficiently relaxed in the vicinity of these critical points to permit the approximant to conform and still remain acceptable.

The bulk of this study deals with prescribed functions which are positive and finite for all frequencies except that they may have zeros at zero and infinity. A method is given in Chapter VI for removing poles and zeros occurring at finite non-zero frequencies. In the infrequent cases where this situation arises, the method can be applied to remove these singularities and convert the prescribed function to the more common form.

Of course once a satisfactory approximant, say $G_{on}(\omega^2)$, has been obtained, the problem of synthesizing the network still remains. The variable ω^2 is replaced by $-s^2$, and appropriate synthesis procedures are applied. Since $G_{on}(-s^2)$ is a "magnitude squared" function, its poles and zeros have quadrantal symmetry. If G_{on} represents the squared magnitude of a transfer function, that is,

$$G_{on}(-s^2) = T_{12}(s)\ T_{12}(-s)$$

such questions as whether the transfer function is to be minimum-phase or non-minimum-phase must be decided in the course of assigning the zeros of $G_{on}(-s^2)$ to $T_{12}(s)$ or $T_{12}(-s)$. These matters, however, have reference to the realization problem in the network synthesis procedure and not to the approximation problem treated in this study.

<u>Expansion in continued fraction form</u>.--The concept of the approximation method is simple. Let $G_{on}(\omega^2)$ be a rational function which represents an acceptable approximation to $G_o(\omega^2)$. Let $G_{on}(\omega^2)$ be expanded in the following form.

$$G_{on}(\omega^2) = \cfrac{1}{a_o(\omega^2) + \cfrac{1}{a_1(\omega^2) + \cfrac{1}{\ddots + \cfrac{1}{a_n(\omega^2)}}}} \tag{1}$$

The terms $a_k(\omega^2)$ are rational functions, and are not necessarily positive for all values of ω. Suppose that $G_o(\omega^2)$ were expanded as follows.

$$G_o(\omega^2) = \cfrac{1}{a_o(\omega^2) + \cfrac{1}{a_1(\omega^2) + \cfrac{1}{\ddots + \cfrac{1}{a_n(\omega^2) + \cfrac{1}{a_{n+1}(\omega^2) + \ddots}}}}} \tag{2}$$

Then $G_{on}(\omega^2)$ is obtained by omitting all after the term $a_n(\omega^2)$ in the expansion (2) of $G_o(\omega^2)$ above. For brevity the description of the variable will be omitted henceforth from the terms G_o, G_{on}, a_k, and related terms to be introduced later, it being understood that these are always functions of ω^2.

The expansion of G_o in (2) above is a continued fraction. Following the accepted terminology of continued fraction theory (17), the terms a_k are called elements, and G_{on} is the nth approximant to the continued fraction expansion of G_o. Thus

$G_{oo} = \frac{1}{a_o}$ is the zeroth approximant to G_o,

$G_{o1} = \cfrac{1}{a_o + \cfrac{1}{a_1}}$ is the first approximant to G_o, and so forth. A very useful term is defined by the following equation:

$$G_k = \frac{1}{a_k + G_{k+1}} \tag{3}$$

G_k (except for G_o) are called remainders.

The essence of the method is to select successive elements, a_k, in accordance with some strategy or policy based on the known remainders, G_k, to produce succeeding remainders, G_{k+1}, in such a manner that eventually some G_{n+1} may be discarded, leaving G_{on} as the desired acceptable approximant. Several expansion strategies are considered in this study.

A choice of expansions.--The expansion (2) of G_o is not unique but depends upon the particular strategy adopted for the selection of a_k. To demonstrate this fact, the way in which six different strategies could produce the same approximant is illustrated in the examples below. Let the rational function $\frac{\omega^4 + 3\omega^2 + 1}{\omega^6 + 5\omega^2 + 2}$ be the acceptable approximant, G_{on}, to a given G_o. Underneath the statement of the strategy in each paragraph the corresponding expansion of G_o through the element a_n is given. The quantities b_k are constants, and p_k are integers which may be positive, negative, or zero. The six strategies and corresponding expansions follow:

(1) The a_k are chosen so that $a_k G_k$ are as small as is feasible but equal to or greater than unity for all ω.

$$G_o = \cfrac{1}{\omega^2 + 2 + \cfrac{1}{-\dfrac{.20\,\omega^2 + .52}{\omega^2} + \cfrac{1}{125\,\omega^4 + 50\,\omega^2 + \ddots}}}$$

(2) The a_k are chosen so that a_kG_k are as large as is feasible but equal to or less than unity for all ω.

$$G_o = \cfrac{1}{\cfrac{2\omega^4 - \omega^2 + 10}{2\omega^2 + 5} + \cfrac{1}{2\omega^2 + 11 + \cfrac{1}{\cfrac{\omega^4}{17\omega^2 + 5} + \cdots}}}$$

(3) The a_k are chosen to have the form $b_k\,\omega^{2p_k}$ and the property:

$$\lim_{\omega \to \infty} a_kG_k = 1$$

$$G_o = \cfrac{1}{\omega^2 + \cfrac{1}{-\frac{1}{3} + \cfrac{1}{-\frac{9\omega^2}{13} + \cfrac{1}{\frac{169}{201} + \cfrac{1}{-\frac{4489\omega^2}{13} + \cfrac{1}{-\frac{1}{134} + \cdots}}}}}}$$

(4) The a_k are chosen to have the form $b_k\,\omega^{2p_k}$ and the property:

$$\lim_{\omega \to o} a_kG_k = 1$$

$$G_o = \cfrac{1}{2 + \cfrac{1}{-\frac{1}{\omega^2} + \cfrac{1}{-1 + \cfrac{1}{\frac{1}{\omega^4} + \cfrac{1}{\frac{\omega^2}{2} + \ldots}}}}}$$

(5) The a_k are chosen to have the form $b_k \left(\frac{\omega^2}{\omega_o^2} - 1\right)^{p_k}$ and the property:

$$\lim_{\omega \to \omega_o} a_k G_k = 1$$

$$G_o = \cfrac{1}{\frac{8}{5} + \cfrac{1}{\frac{25}{7(\omega^2-1)^2} + \cfrac{1}{\frac{49(\omega^2-1)}{50} + \cfrac{1}{\frac{500}{7(\omega^2-1)^2} + \cfrac{1}{\frac{\omega^2-1}{50} + \ldots}}}}} \qquad (\text{for } \omega_o = 1)$$

(6) The a_k are chosen to have the form $(b_{k1}\omega^{2p_{k1}} + b_{k2}\omega^{2p_{k2}})^{\pm 1}$ and the property:

$$\lim_{\omega \to \infty} a_k G_k = 1 \quad \text{and} \quad \lim_{\omega \to 0} a_k G_k = 1$$

$$G_o = \cfrac{1}{\omega^2 + 2 + \cfrac{1}{-\frac{1}{5} - \frac{1}{2\omega^2} + \cfrac{1}{-50\omega^2 - 20 + \ldots}}}$$

The various expansion strategies mentioned above are considered in more detail in subsequent chapters.

Logarithmic calculations.--The calculations involved in the method may be performed simply in logarithmic terms. Let

$$H_k = 10 \log G_k$$

$$A_k = 10 \log a_k$$

Logarithms are to the base 10, and the units of H_k and A_k are therefore decibels. The equation (3) relating successive remainders yields:

$$G_{k+1} = \frac{1}{G_k} - a_k \qquad (4)$$

$$G_{k+1} = \frac{1}{G_k}(1 - a_k G_k) \qquad (5)$$

$$H_{k+1} = - H_k + 10 \log(1 - \text{antilog} \frac{1}{10}(A_k + H_k)) \qquad (6)$$

The following very useful function is introduced.

$$L(u) = 10 \log(1 - \text{antilog} \frac{u}{10}) = 10 \log(1 - 10^{\frac{u}{10}}) \qquad (7)$$

Making use of the L-function,

$$H_{k+1} = - H_k + L(A_k + H_k) \qquad (8)$$

The properties of the function $L(u)$ and a tabulation are given in Appendix C. Note that when u is a positive real number $L(u)$ is complex. Thus

$$L(4.7) = 10\ \log(10^{.47} - 1) + 10\ \log(-1) = 2.90 + \bar{I}$$

The symbol $\bar{I}$ represents 10 log(-1) and has the value

$$\bar{I} = 10\ \log(-1) = \pm \frac{j10\pi}{\ln 10}$$

Thus $\bar{I}$ is a pure imaginary number. This suggested the selection of the capital letter I for its symbol, and a bar is added to avoid any possible ambiguity. The symbol $\bar{I}$ is called "eye-bar". $\bar{I}$ does not enter numerically into the calculations and may therefore be carried along in symbolic form. It serves merely as a reminder of the proper sign to be used when an anti-logarithm is to be taken. The sign associated with $\bar{I}$ is immaterial and the positive sign is used throughout this paper. When $\bar{I}$ appears an even number of times in a sum it is discarded. Thus

$$u + \bar{I} + v + \bar{I} = u + v$$

A few of the important properties of L(u) are:

$$L(u + \bar{I}) = 10\ \log(1 + 10^{\frac{u}{10}}) \tag{9}$$

$$L\left\{L(u)\right\} = u \tag{10}$$

$$L(-u) = L(u) - u + \bar{I} \tag{11}$$

Combining equation (11) with equation (8),

$$H_{k+1} = - H_k + L(A_k + H_k) = \bar{I} + A_k + L(-A_k - H_k) \tag{12}$$

Both expressions for H_{k+1} in (12) are equivalent; the second one often makes calculations simpler when A_k is a straight line.

Equation (10) may be applied to equation (8) to derive

$$H_k = - A_k - L(- A_k + \bar{I} + H_{k+1}) \tag{13}$$

Repeated application of (13) permits calculation of H_{on} at any stage of the procedure. Thus, for example, the equation for the first approximant,

$$G_{o1} = \frac{1}{a_o + \frac{1}{a_1}}$$

becomes, in logarithmic terms,

$$H_{o1} = - A_o - L(- A_o + \bar{I} - A_1).$$

The more general equation for H_{on} is:

$$\begin{aligned} H_{on} = 10 \log G_{on} = - A_o - L(- A_o + \bar{I} - A_1 \\ - L(- A_1 + \bar{I} - A_2 - L(\ldots\ldots - L(- A_{n-1} \\ + \bar{I} - A_n) \ldots\ldots))). \end{aligned} \tag{14}$$

The calculations of the proposed semi-graphical method are based on the tabulated values of H_o; if H_o is given only in graphical form a tabulation is first made from the graph. Nevertheless graphical sketches of the various H_k are essential as an aid in selecting the successive A_k. H_k and A_k are therefore plotted against the usual logarithmic frequency scale, $v = \log \omega$, for abscissas. Example 1, Appendix A, demonstrates the mechanics of the logarithmic calculations.

Illustrative example.--A simple example will serve to outline the general procedure followed in the proposed method and to illustrate the relation between algebraic and logarithmic quantities. In Figure 1 a prescribed function, H_o, is sketched; the magnitude of H_o is expressed in decibels. In Figure 2 the prescribed function is portrayed as G_o; it is the same function as H_o, but its magnitude is expressed in algebraic units. The frequency scale is logarithmic in both figures; v equals log ω. The first element in the continued fraction expansion of G_o is chosen to be the constant 4.00; its reciprocal is plotted on Figure 2, and the corresponding logarithmic quantity, $-A_o$, is drawn on Figure 1. The value of $-A_o$ is -6.02 decibels.

From equation (4) the first remainder, G_1, is calculated point by point from the given values of G_o and the appropriate values of the element a_o just selected. For example, in this problem for v equals zero (ω equals unity), G_o equals 0.414 and a_o equals 4.00. Therefore,

$$G_1 = \frac{1}{G_o} - a_o = \frac{1}{0.414} - 4 = -1.585$$

G_1 is sketched in Figure 2. In this example the algebraic values are shown only to demonstrate the relation between algebraic and logarithmic quantities. In the proposed method the calculations are actually performed with the logarithmic quantities. H_1 is calculated using equation (8) and the tables of L(u). Thus for v equals zero, H_o is given as -3.83, A_o equals 6.02, and

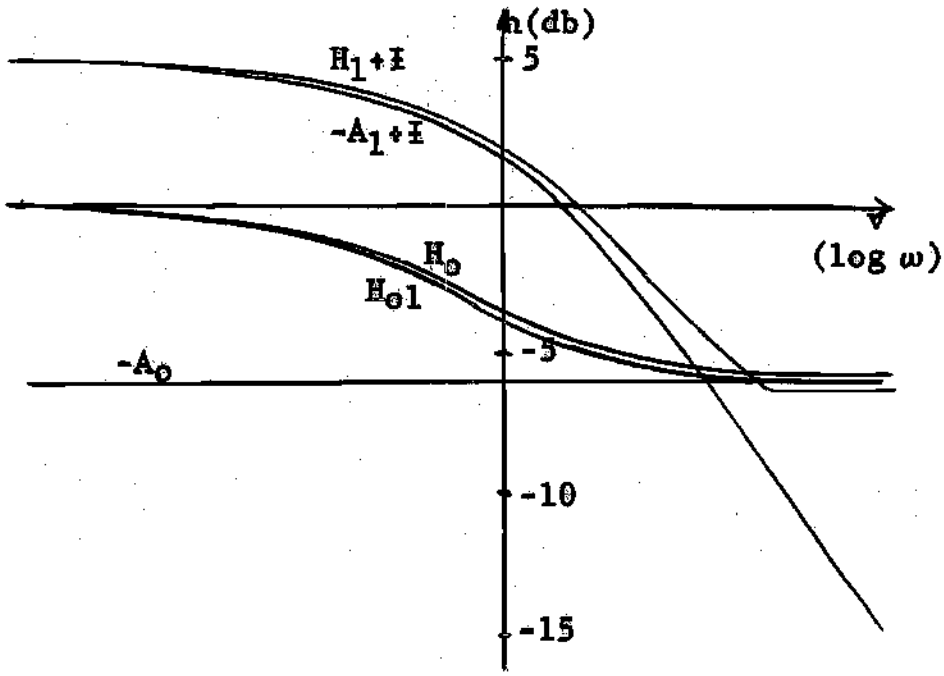

Fig. 1. Sketch to Logarithmic Ordinates of Quantities from the Illustrative Example.

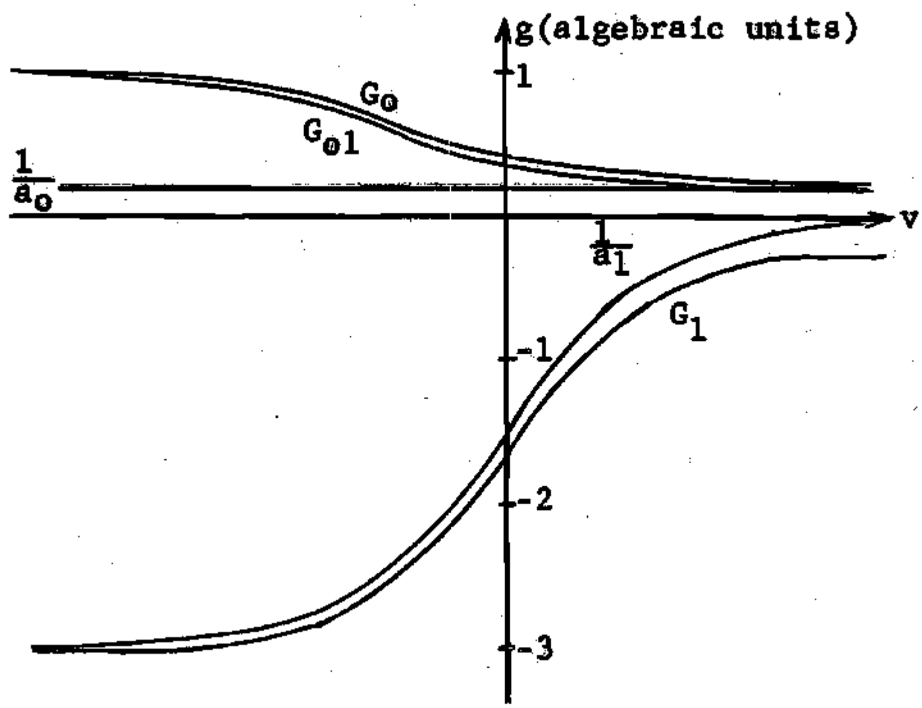

Fig. 2. Sketch to Algebraic Ordinates of Quantities from the Illustrative Example.

$$
\begin{aligned}
H_1 &= -H_o + L(H_o + A_o) \\
&= 3.83 + L(2.19) \\
&= 2.00 + \text{Ɨ}
\end{aligned}
$$

The symbol Ɨ is present because G_1 is negative. H_1 + Ɨ (the real part of H_1) is sketched in Figure 1. The first stage in the continued fraction expansion is thus completed.

The next step is to choose another element; $-A_1$ is chosen equal to $4.77 - 10 \log(1 + \omega^2) + \text{Ɨ}$, and is plotted on Figure 1. The corresponding expression for the algebraic element a_1 is $-\dfrac{\omega^2 + 1}{3}$, since a_1 is the antilogarithm of one-tenth of A_1. The reciprocal of a_1 is sketched in Figure 2.

The expansion is terminated at this point. The final approximant which has been generated is

$$
G_{o1} = \frac{1}{a_o + \frac{1}{a_1}} = \frac{1}{4 - \frac{1}{\frac{1}{3}(\omega^2 + 1)}} = \frac{\omega^2 + 1}{4\omega^2 + 1}
$$

G_{o1} is sketched on Figure 2 and is evidently a good approximant to G_o. Its logarithmic expression, H_{o1}, may be calculated point by point by using the equation

$$
H_{o1} = -A_o - L(-A_o + \text{Ɨ} - A_1)
$$

For example, at v equals zero,

$$H_{o1} = -6.02 - L(-6.02 + \mathrm{I} + 1.76 + \mathrm{I})$$

$$= -6.02 - L(-4.26)$$

$$= -3.98$$

H_{o1} is sketched on Figure 1.

<u>Behavior of continued fractions</u>.--Before beginning a full description of the proposed semi-graphical method it would be valuable to have at hand some relationships that can be gleaned from the theory of continued fractions. For example, these relationships may assure us, providing the conditions for their existence are satisfied, that some of the following properties hold true at a certain frequency or in a certain range of frequencies:

(1) A higher-ordered approximant is better than a lower-ordered one. This was true of the example of the preceding paragraph, in that G_{o1} was closer to G_o than was G_{oo} (G_{oo} is equal to $1/a_o$).

(2) A certain approximant to the prescribed function is closer to the prescribed function than the last element is to the remainder which that element approximates. In the preceding example this holds true for v equal to or greater than -0.1; Thus for v equals zero

$$H_o - H_{o1} = 0.15 \text{ db.} \qquad H_1 - (-A_1) = 0.24 \text{ db.}$$

or, in algebraic terms,

$$1 > \frac{G_{o1}}{G_o} = \frac{0.400}{0.414} = 0.967 > \frac{1/a_1}{G_1} = \frac{-1.500}{-1.585} = 0.946$$

(3) The difference between a certain approximant and the prescribed function (in logarithmic terms) is less than a value readily obtained from certain tables (not illustrated in the preceding example).

(4) The reciprocal of an element ought to be chosen to approximate closely its corresponding remainder in one segment of the frequency spectrum, whereas in another segment it may be permitted to diverge from the remainder by large values. In the preceding example $-A_1$ was chosen to be very close to H_1 at low frequencies; at high frequencies it was permitted to diverge from H_1, yet the final approximant H_{o1} was very good at all frequencies.

These useful relationships and others are therefore developed in Chapter III prior to full exposition of the proposed method. Subsequently Chapters IV and V advance the principal variations of the proposed semi-graphical method. Chapter VI deals with some additional variations of less utility, and Chapter VII examines the accuracy of the method.

CHAPTER III

CONVERGENCE OF APPROXIMANTS

<u>Relation between continued fractions and a suitable strategy for the selection of elements</u>.--Reference has been made in the preceding chapter to a strategy in accord with which the selection of successive elements a_k is to be made. The object of this strategy is to select the elements in such a way that the magnitude of the difference between an acceptable approximant and the logarithmic prescribed function is less than some permissible tolerance limit which is also prescribed. Thus where t is the tolerance limit in decibels, the fact that

$$H_{on} - H_o \leqq t \text{ for all values of } \omega$$

indicates that H_{on} is an acceptable approximant. In the algebraic domain the above inequality becomes

$$1 \leqq \left(\frac{G_{on}}{G_o}\right)^{\pm 1} \leqq \epsilon$$

where the sign of the exponent is chosen to make the left inequality hold, and ϵ, being the antilogarithm of one-tenth of t, is greater than unity, usually by a small amount.

The inequality above indicates that a suitable strategy ought to insure that G_{on}/G_o is positive and close to unity. If for a particular value of n, G_{on}/G_o is not close enough to unity, the expansion in continued fraction form is extended another stage, and $G_{o,n+1}/G_o$ is now examined to see whether or not it is acceptably close to unity. Thus another desirable property of an expansion strategy is that G_{on}/G_o is closer to unity than G_{ok}/G_o for n greater than k, or, in other words, that higher-ordered approximants are better than lower-ordered ones. Evidently the relation between G_{on}/G_o and G_{nn}/G_n (which equals $1/a_nG_n$) is also of interest as a guide to the selection of a_n. Of course, even if G_{on}/G_o draws closer to unity as the order n increases, it may not converge to unity, or even to a value less than ε, and the question of convergence of the series of approximants to G_o is therefore also a matter with which a suitable strategy is concerned.

This chapter draws on the theory of continued fractions to present an investigation of the relationships between successive approximants to the prescribed function, between approximants to remainders and approximants to the prescribed function, and between the prescribed function and the value, if it exists, of its continued fraction expansion. The chapter also deals with relations between the series of rational function approximants and comparison series of approximants to the same prescribed function, the comparison series having a specified relationship between each element and its corresponding remainder. Practical questions regarding the source of the element functions and the detailed application of the results of this chapter to the logarithmic method of calculation are deferred to later chapters. The results of this chapter are obtained in the algebraic domain, but are easily transferred to the logarithmic domain for use in subsequent chapters.

<u>Terminology</u>.--The symbols G_o, a_k, G_k, and G_{on} have already been introduced. G_o is the prescribed function which it is desired to approximate. The k^{th} element in the continued fraction expansion of G_o is a_k, and is a rational function of ω^2. Depending upon the particular strategy being employed a_k may be drawn from the class of general rational functions or may be restricted to a particular form of rational function. G_k indicates the k^{th} remainder in the continued fraction expansion of G_o. Thus, for example, G_4 is defined by the following equation:

$$G_o = \cfrac{1}{a_o + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + G_4}}}}$$

From the above it is easily seen that the basic recurrence relation between successive remainders is:

$$G_k = \frac{1}{a_k + G_{k+1}} \tag{15}$$

Since G_o is a tabulated function it has no analytic expression. Therefore, even though the elements a_k are analytic functions of ω^2, the remainders G_k, which depend on G_o as well as on the elements, are also merely tabulated functions having no known analytic expression.

G_{on} is the n^{th} approximant to G_o, and is obtained by replacing the remainder G_{n+1} with zero in the continued fraction expansion. Another way of looking at it is that G_{on} is a truncated continued fraction expansion obtained by discarding everything after the element a_n. Thus

$$G_{o4} = \cfrac{1}{a_o + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4}}}}}$$

Evidently G_{oo} and $1/a_o$ are identical. G_{on} is a rational function of ω^2.

Symmetry suggests the introduction of the term G_{kn}. G_{kn} is the $n-k^{th}$ approximant to the remainder G_k. G_{kn} bears the same relation to G_k as G_{on} bears to G_o. For example

$$G_{24} = \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4}}}$$

It follows that

$$G_{o4} = \cfrac{1}{a_o + \cfrac{1}{a_1 + G_{24}}}$$

G_{kk} and $1/a_k$ are also identical, G_{kk} being the zeroth approximant to the remainder G_k. Of course G_{kn} has no meaning if n is less than k. G_{kn} is also a rational function of ω^2. In a manner analogous to equation (15),

$$G_{kn} = \frac{1}{a_k + G_{k+1,n}} \tag{16}$$

In dealing with questions of continued fraction convergence a symbol is needed to represent the continued fraction expansion of G_o, and to distinguish it from the given tabulated function G_o. The symbol $G_{o\infty}$ is chosen to represent the continued fraction expansion of G_o. Thus

$$G_{o\infty} = \cfrac{1}{a_o + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}}$$

If the above continued fraction is not convergent no value is assigned to $G_{o\infty}$; it is merely used to indicate the form of the right-hand member. If the continued fraction is convergent, the limit of the series of approximants is designated G_o^*, and

$$G_{o\infty} = G_o^*$$

It must be pointed out that G_o^*, even if it exists, may or may not be equal to G_o.

The similarity between equations (15) and (16) suggests the introduction of the general recurrence relation,

$$g_k = \frac{1}{a_k + g_{k+1}} \tag{17}$$

In equation (17) g_k denotes the range of values of which G_k and G_{kn} are particular selections. In other words g_k identifies the axis on which G_k and the various G_{kn} (for different values of n) are points; g_k may be called the k^{th}-stage axis. In the course of the derivations of this chapter it is also useful to be able to designate arbitrarily selected points on the k^{th}-stage axis which are not specifically the k^{th} remainder or any particular approximant to the k^{th} remainder; ${}_1g_k$ and ${}_2g_k$ will serve to identify such points. All points on the k^{th}-stage axis are related to corresponding points on the $k+1^{st}$-stage axis by the basic recurrence equation (17), as demonstrated by equations (15) and (16) and the examples below.

$$ {}_2g_k = \frac{1}{a_k + {}_2g_{k+1}} \qquad\qquad G_{27} = \frac{1}{a_2 + G_{37}} $$

Equation (17) is an hyperbola, as indicated by the plot of a typical curve in Figure 3. Some of the terms described above are also illustrated in the figure.

<u>The role of frequency in the continued fraction expansion of tabulated functions</u>.--A further explanation of the significance of figures such as Figure 3 must include the relationship of frequency to these figures, and for this we return to a further examination of the ratio G_{on}/G_o previously discussed in the first section of this chapter. By its construction G_{on} represents a rational function of ω^2. G_o, however, is merely a tabulated function. Consequently the ratio G_{on}/G_o is also only a tabulated function and has no analytic expression. As a result when we examine the ratios G_{on}/G_o for increasing n to determine questions of convergence and relationships between successive approximants, we are restricted to comparisons at a specific frequency. The fact that G_{on}/G_o is acceptably close to unity at one frequency says nothing about its behavior at another frequency. Essentially questions of convergence and acceptability of approximants represent problems to be solved separately for each frequency in the spectrum of interest. The same reasoning applies to G_{kn}/G_k, of which G_{on}/G_o is the particular case for k equal to zero.

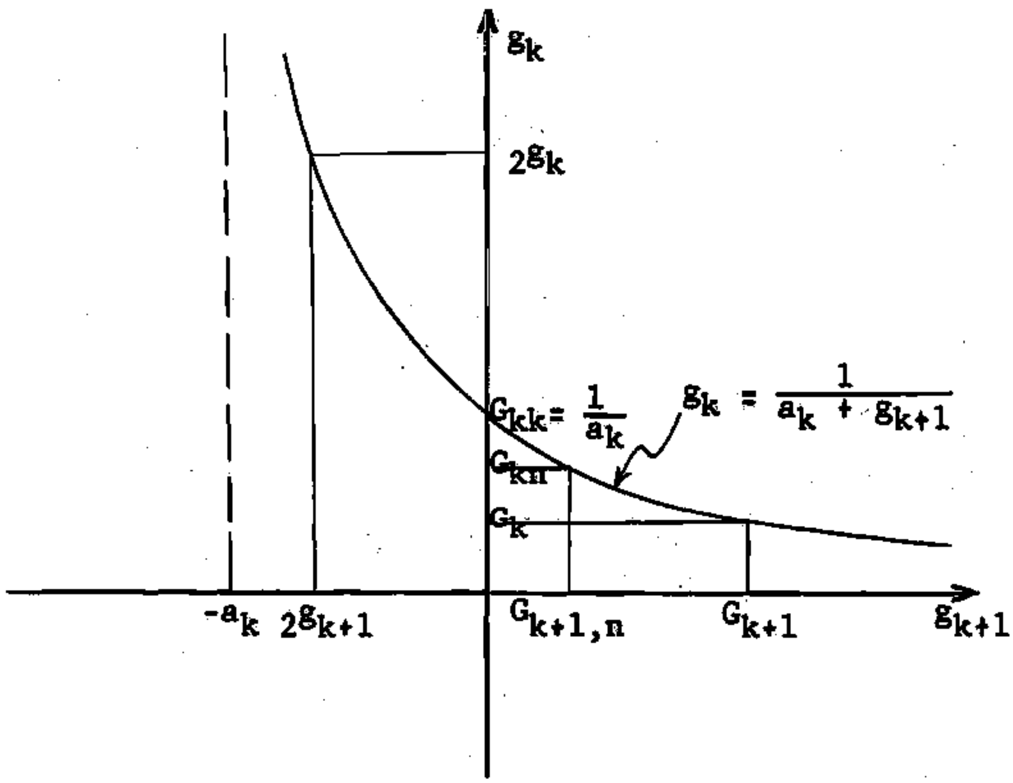

Fig. 3. Terminology.

Points on both axes in Figure 3 represent values of the respective functions at the same frequency. The symbols are explained in the text. The figure illustrates the relation between quantities on the g_k axis and corresponding quantities on the g_{k+1} axis. The dotted line, $g_{k+1} = -a_k$, is an asymptote of the hyperbola. The curve in the figure is plotted for the case $a_k G_k < 1$.

It follows that in Figure 3 and others of the same type G_k represents the value of the k^{th} remainder at a specific frequency, and the other quantities indicated have their values at the same frequency. However this does not restrict the utility of the figures as much as might first be supposed, because the chief points of interest are the inequalities indicated by the figures. If the figures are regarded as sketches illustrating the pertinent inequalities (such as whether G_{kn} is larger or smaller than G_k) rather than as precise plots of specific values, then they may have validity over a range of frequencies or even over the entire frequency spectrum of interest.

A definition of "better" is needed with which to compare approximants. G_{on} is said to be a better approximant to G_o than G_{ok} if

$$\left(\frac{G_{ok}}{G_o}\right)^{\pm 1} > \left(\frac{G_{on}}{G_o}\right)^{\pm 1} > 1$$

the signs of the exponents being chosen to make the ratios greater than unity. This is equivalent, of course, to stating that H_{on} is better than H_{ok} if the magnitude of the difference in decibels between H_{on} and H_o is less than that between H_{ok} and H_o.

<u>Properties of elements</u>.--In Chapter II several strategies for expanding G_o in continued fraction form were noted. In two of these the elements a_k were permitted to be general rational functions, while in the others a_k were rational functions of specified form. In this chapter the elements will initially be considered to be general rational functions subject to

the restriction that every product $a_k G_k$ be positive. In other words each element will be chosen to have the same sign as its corresponding remainder at all frequencies.

The behavior of successive approximants divides their analysis into two cases, which will be treated concurrently in this chapter. In one case all products $a_k G_k$ are assumed to be less than unity, and in the other, greater than unity. In the former case, therefore, $1/a_k$ are chosen to be greater than G_k (as illustrated in Figure 3), while in the latter case $1/a_k$ are chosen to be less than G_k. Of course ideally $1/a_k$ should be chosen equal to G_k; if this could be accomplished at all frequencies for some particular a_k, say a_n, then the remainder G_{n+1} would be identically zero and G_{on} would be exactly equal to G_o. However G_k is a tabulated function, while $1/a_k$ is restricted to be a rational function, and it would therefore be an exceptional case if any $1/a_k$ could be chosen to be exactly equal to G_k at all frequencies. Consequently the best that can be hoped for is that $1/a_k$ will be chosen as close to G_k as is feasible with a rational function of limited complexity. By requiring one or the other of the two inequality restrictions cited above the behavior of continued fractions can be utilized to develop some useful properties of the expansions.

The effects produced when any $a_k G_k$ equals unity at an isolated frequency, when $a_k G_k$ do not follow the same inequality rule in successive stages, or when any $a_k G_k$ does not follow the same inequality rule throughout the frequency spectrum of interest, all three being interrelated questions, are considered at the end of the chapter.

Relations between approximants and corresponding remainders.--Equation (5), reproduced below, is the equation used to obtain the remainder G_{k+1} from the remainder G_k.

$$G_{k+1} = \frac{1}{G_k}(1 - a_k G_k)$$

It may be combined with equation (16) to yield

$$\frac{G_{kn}}{G_k} = \frac{1}{1 + (a_k G_k - 1)(1 - \frac{G_{k+1,n}}{G_{k+1}})} \tag{18}$$

The two cases identified in the preceding section are treated consecutively.

(1) $a_k G_k < 1$: Equation (5) shows that G_k and G_{k+1} have the same sign. Since G_o is positive, all G_k are positive; and since all products $a_k G_k$ are positive, as explained in the preceding section, all a_k are positive. G_{nn} is therefore positive. Equation (16) shows that G_{kn} has the same sign as $G_{k+1,n}$, so by induction all G_{kn} are positive.

Equation (18) shows that if $G_{k+1,n}/G_{k+1}$ is greater than or less than one, G_{kn}/G_k is respectively less than or greater than one. Since G_{nn}/G_n equals $1/a_n G_n$ and is therefore greater than one, G_{kn}/G_k is greater than unity for n-k even and less than unity for n-k odd.

(2) $a_k G_k > 1$: Equation (5) shows that G_k and G_{k+1} have opposite signs. Since G_o is positive and a_k have the same signs as the corresponding G_k, G_k and a_k are positive for k even and negative for k odd.

In this case equation (18) shows that if $G_{k+1,n}/G_{k+1}$ is positive and less than unity, then G_{kn}/G_k is also positive and less than unity. Since G_{nn}/G_n has this property, all G_{kn}/G_k do likewise.

Recapitulating for both cases,

$$\text{if} \quad a_k G_k < 1, \; G_k > 0 \quad \text{and} \quad \left(\frac{G_{kn}}{G_k}\right)^{(-1)^{n-k}} > 1 \tag{19}$$

$$\text{and if} \quad a_k G_k > 1, \; (-1)^k G_k > 0 \quad \text{and} \quad 0 < \frac{G_{kn}}{G_k} < 1 \tag{20}$$

<u>A relation between different approximants to the same remainder</u>.--Let ${}_1g_k$ and ${}_2g_k$ be arbitrary points on the k^{th}-stage axis. They are related by equation (17) to corresponding points on the $k+1^{st}$-stage axis as follows.

$${}_1g_k = \frac{1}{a_k + {}_1g_{k+1}} \qquad {}_2g_k = \frac{1}{a_k + {}_2g_{k+1}}$$

The above equations can be combined with equation (5) to yield:

$$\frac{\dfrac{{}_1g_k}{G_k}}{\dfrac{{}_2g_k}{G_k}} = \frac{\dfrac{{}_2g_{k+1}}{G_{k+1}}\left(-1 + \dfrac{1}{a_k G_k}\right)}{1 + \dfrac{{}_1g_{k+1}}{G_{k+1}}\left(-1 + \dfrac{1}{a_k G_k}\right)} \tag{21}$$

The two cases of interest are again taken up in order.

(1) $a_k G_k < 1$: From equation (16)

$$\frac{G_{p,p+2}}{G_{pp}} = a_p G_{p,p+2} = \frac{1}{1 + \dfrac{G_{p+1,p+2}}{a_p}}$$

In view of (19), and since a_p and $G_{p+1,p+2}$ are positive in this case,

$$\frac{G_{pp}}{G_p} > \frac{G_{p,p+2}}{G_p} > 1$$

It being observed that in this case the expressions in parentheses in equation (21) are positive, it may be stated that if $\frac{{}_1g_{k+1}}{G_{k+1}} \lesseqgtr \frac{{}_2g_{k+1}}{G_{k+1}}$, then $\frac{{}_1g_k}{G_k} \gtreqless \frac{{}_2g_k}{G_k}$. This result may be extended by repeated application to yield

$$\frac{{}_1g_p}{G_p} > \frac{{}_2g_p}{G_p} \text{ implies } \begin{cases} \frac{{}_1g_k}{G_k} > \frac{{}_2g_k}{G_k} \text{ (p-k even, } p > k) \\ \\ \frac{{}_1g_k}{G_k} < \frac{{}_2g_k}{G_k} \text{ (p-k odd, } p > k) \end{cases}$$

Substituting G_{pp} for ${}_1g_p$ and $G_{p,p+2}$ for ${}_2g_p$ in the above inequalities, it follows that

$$\frac{G_{kp}}{G_k} > \frac{G_{k,p+2}}{G_k} > 1 \quad \text{for p-k even}$$

$$\frac{G_{kp}}{G_k} < \frac{G_{k,p+2}}{G_k} < 1 \text{ for p-k odd}$$

This may be generalized by induction to the useful result:

$$\left(\frac{G_{kp}}{G_k}\right)^{(-1)^{p-k}} > \left(\frac{G_{kn}}{G_k}\right)^{(-1)^{n-k}} > 1 \quad \text{for n-p even} \qquad (22)$$

$$(n > p \geq k)$$

Note that the result only holds for n-p even.

(2) $a_k G_k > 1$: The following equation is easily derived.

$$\frac{G_{p,p+1}}{G_{pp}} = \frac{1}{1 + \frac{1}{a_p a_{p+1}}}$$

In this case a_p and a_{p+1} have different signs, so that from the above and (20),

$$\frac{G_{pp}}{G_p} < \frac{G_{p,p+1}}{G_p} < 1$$

Examination of equation (21) in this case shows that the expressions in parentheses are negative but smaller in magnitude than unity. By an analysis similar to that of the preceding paragraph we obtain

$$\frac{{}_1g_p}{G_p} < \frac{{}_2g_p}{G_p} < 1 \quad \text{implies} \quad \frac{{}_1g_k}{G_k} < \frac{{}_2g_k}{G_k} < 1 \ (p > k)$$

Substituting G_{pp} for ${}_1g_p$ and $G_{p,p+1}$ for ${}_2g_p$ in the above inequalities, we obtain the result

$$\frac{G_{kp}}{G_k} < \frac{G_{k,p+1}}{G_k} < 1$$

This last is readily extended by induction to

$$\frac{G_{kp}}{G_k} < \frac{G_{kn}}{G_k} < 1 \qquad \text{for } n > p \geq k \tag{23}$$

The results of (22) and (23) may be summarized as follows. As the prescribed function, or any remainder, is expanded in continued fraction form, creating a series of successive approximants, the approximants behave in one of the following two alternative manners. If the strategy used insures that all products $a_k G_k$ are less than unity, every approximant is better than the one preceding it by two terms in the series. On the other hand, if all $a_k G_k$ are greater than unity, every approximant is better than the one immediately preceding it. This behavior is illustrated in Figures 4 and 5.

<u>A relation between corresponding approximants to different remainders.</u>-- The term corresponding here refers to approximants to different remainders achieved with expansions having the same last element. Thus G_{pn} and G_{kn} are corresponding approximants to G_p and G_k. For p greater than k, of course, G_{kn} is a function of G_{pn} and the elements from a_k to a_{p-1} inclusive.

(1) $a_k G_k < 1$: Equation (18) may be rearranged into the form

$$\frac{G_{kn}}{G_k} = \left(\frac{G_{k+1}}{G_{k+1,n}}\right) \frac{1}{1 + a_k G_k\left(\frac{G_{k+1}}{G_{k+1,n}} - 1\right)} \tag{24}$$

Inspection of (24) shows that

$$\frac{G_{k+1,n}}{G_{k+1}} \gtrless 1 \quad \text{implies} \quad \frac{G_{k+1}}{G_{k+1,n}} \gtrless \frac{G_{kn}}{G_k}$$

Combining with (19) and generalizing,

$$\left(\frac{G_{pn}}{G_p}\right)^{(-1)^{n-p}} > \left(\frac{G_{kn}}{G_k}\right)^{(-1)^{n-k}} > 1 \qquad (n \geqq p > k) \tag{25}$$

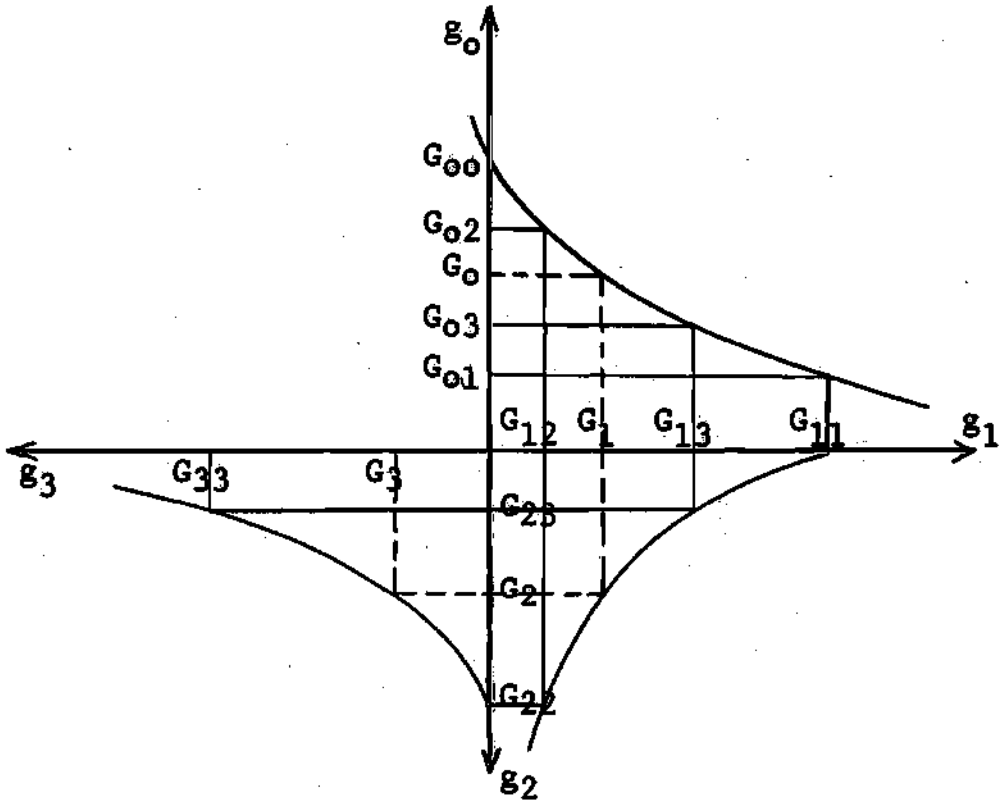

Fig. 4. Improvement of Approximants for a_kG_k Less than Unity.

Figure 4 is a composite figure, in that the curve of equation (17) for successive stages is plotted in different quadrants. In the upper right quadrant is the curve relating g_o to g_1; in the next quadrant clockwise is the curve relating g_1 to g_2, and so forth. This manner of drawing the figure permits projection of quantities from one axis to the related points on all preceding axes in a single diagram. The figure illustrates how the selection of all $1/a_k$ ($= G_{kk}$) greater than G_k produces a series of approximants G_{on}, which are alternately greater than or less than the prescribed function G_o, and each of which is better than the second preceding approximant.

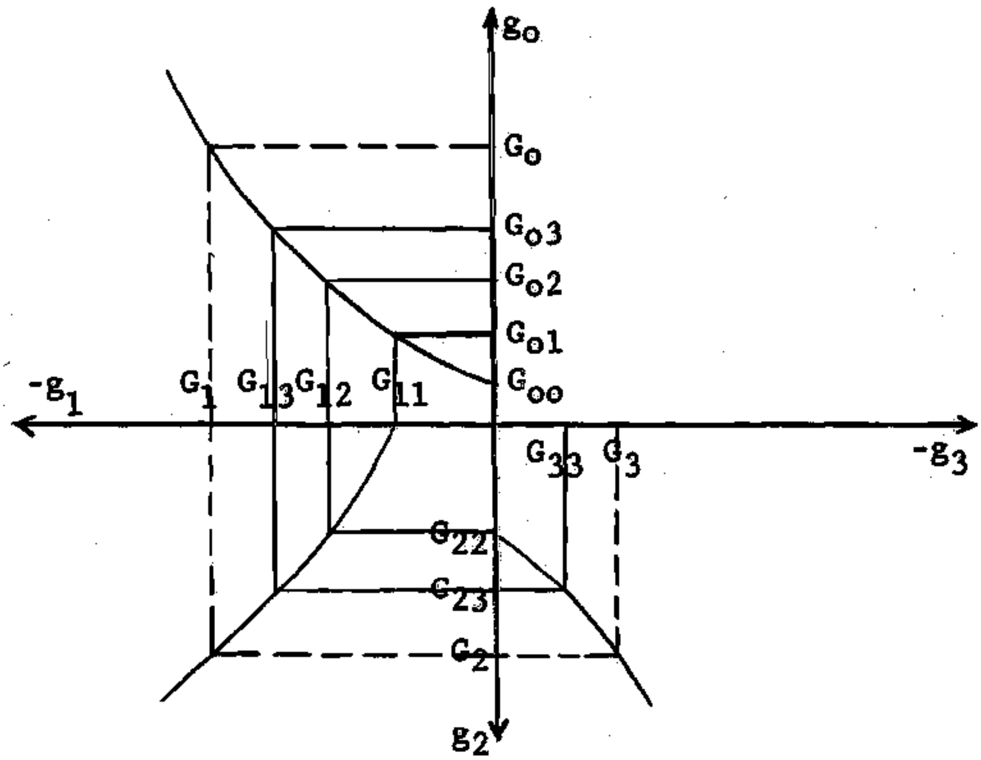

Fig. 5. Improvement of Approximants for $a_k G_k$ Greater than Unity.

Figure 5 is a composite figure in that the curve of equation (17) is plotted for successive stages in different quadrants. In the upper left quadrant is the curve relating g_o to g_1; in the next quadrant counterclockwise is the curve relating g_1 to g_2, and so forth. Note that for the odd-numbered stages the negative part of the g_k axis is the part of interest. The figure illustrates how the selection of all $1/a_k$ ($= G_{kk}$) less than G_k produces a series of approximants G_{on}, each of which is less than the prescribed function G_o, but better than the preceding approximant.

(2) $a_kG_k > 1$: The equation below is merely an identity.

$$\frac{G_{kn}}{G_k} = \frac{G_{k+1,n}}{G_{k+1}} + \frac{G_{kn}}{G_k}\left(1 - \frac{G_{k+1,n}}{G_{k+1}}\ \frac{G_k}{G_{kn}}\right)$$

Substituting from (24) into the expression in parentheses,

$$\frac{G_{kn}}{G_k} = \frac{G_{k+1,n}}{G_{k+1}} + \frac{G_{kn}}{G_k}\left(1 - \frac{G_{k+1,n}}{G_{k+1}}\right)\left(1 + \frac{G_{k+1,n}}{G_{k+1}}(1 - a_kG_k)\right)$$

Since from (20) all $\frac{G_{kn}}{G_k}$ are positive and less than unity, it is evident that if $a_kG_k < 2$, the product on the right is positive and $\frac{G_{k+1,n}}{G_{k+1}}$ is less than $\frac{G_{kn}}{G_k}$. Extending this result through a succession of terms yields:

$$\text{If all } a_kG_k < 2, \quad \frac{G_{pn}}{G_p} < \frac{G_{kn}}{G_k} < 1 \qquad (n \geq p > k) \tag{26}$$

The results of (25) and (26) are most useful for the case when k equals zero and p equals n, and may be summarized as follows. Providing all a_kG_k are less than one, or all fall between one and two, the last approximant is a better approximation to the prescribed function than the reciprocal of the last element is to the last remainder. This relation may be combined advantageously with the relation of the preceding section. For example, if in a particular expansion in which a_kG_k are less than unity it occurs that G_{33} falls closer to G_3 than the prescribed tolerance limits for an acceptable approximant to the given function over a portion of the frequency range, then in succeeding stages of the expansion it is not necessary to attempt to choose elements close to their corresponding remainders in that region. Providing the last element is an odd-numbered one, the final approximant is sure to be acceptable in the range in which G_{33} was.

<u>Convergence of the continued fractions.</u>--The expansion of G_o as a continued fraction has the form

$$G_{o\,\infty} = \cfrac{1}{a_o + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}} \tag{27}$$

Let $\lim_{n \to \infty} G_{on} = G_o^*$, if it exists. The question is raised whether G_o^* exists, and, if it does, whether it is equal to G_o.

In his memoirs on continued fractions Stieltjes (18) investigates the form

$$S(z) = \cfrac{1}{b_1 z + \cfrac{1}{b_2 + \cfrac{1}{b_3 z + \cfrac{1}{b_4 + \ddots}}}} \tag{28}$$

in which the b_k are all real positive numbers, and finds that the question of its convergence is divided into two cases:

(a) If $\lim_{n\to\infty} \sum_1^n b_k$ converges, the even and odd approximants of $S(z)$ converge to different limits (19). If both limits are finite $S(z)$ is said to diverge by oscillation (20).

(b) If $\lim_{n\to\infty} \sum_1^n b$ diverges, the approximants to $S(z)$ converge to a single limit provided z is not on the negative real axis, which is to be regarded as a cut in the complex plane of z(21). However, segments of the negative real axis may also be included in the region of convergence of $S(z)$ if none of the approximants to $S(z)$ have singularities on those segments (22).

The above result may be applied to (27) as follows, the analysis again being divided into two cases with respect to the magnitude of the products $a_k G_k$.

(1) $a_k G_k$ are less than unity: Set z equal to unity and b_k equal to a_k in (28) (which may be done as the a_k are all positive in this case). Then

$$G_{o\infty} = \frac{1}{a_o + S(1)}$$

Therefore G_o^* exists if $\sum a_k$ diverges. A more illuminating form of the expansion (27) is obtained if $a_k G_k$ are set equal to r_k (r_k are less than unity) and G_o is expanded as follows:

$$G_{o\infty} = \cfrac{G_o}{r_o + \cfrac{1}{\cfrac{r_1}{1-r_o} + \cfrac{1}{\cfrac{r_2(1-r_o)}{1-r_1} + \cfrac{1}{\cfrac{r_3(1-r_1)}{(1-r_2)\,(1-r_o)} + \cdots}}}} \qquad (29)$$

The condition for convergence of (29) becomes:

$$\sum_{k=0}^{\infty} (r_k) \frac{\prod_{i=0}^{i\leq\frac{k}{2}-1} (1 - r_{k-2-2i})}{\prod_{j=0}^{j<\frac{k}{2}} (1 - r_{k-1-2j})} \quad \text{diverges.} \tag{30}$$

Special cases of interest include the following:

(a) All r_k are the same and equal r. In this case all even elements equal r and all odd elements equal $\frac{r}{1-r}$. Therefore (30) diverges and G_o^* exists.

(b) The values of r_k repeat after r_{n-1}, that is, $r_n = r_o$, $r_{n+1} = r_1$, and so forth. When the ratio

$$R_e = \frac{a_{2n}}{a_o} = \frac{a_{4n}}{a_{2n}} = \frac{a_{6n}}{a_{4n}} = \ldots\ldots$$

is formed it is found to be the reciprocal of the ratio

$$R_o = \frac{a_{2n+1}}{a_1} = \frac{a_{4n+1}}{a_{2n+1}} = \frac{a_{6n+1}}{a_{4n+1}} = \ldots\ldots$$

One of these ratios must be equal to or greater than unity, with the result that the sum of the even elements or the sum of the odd elements diverges. Thus (30) diverges and G_o^* exists.

In all cases where a_kG_k are less than unity, where G_o^* exists it equals G_o. This follows from (19), which shows that successive approximants are alternately greater and less than G_o.

(2) a_kG_k are greater than unity: In this case a_k are positive for k even and negative for k odd. Substituting $(-1)^k a_k$ for b_k in (28)

$$G_{o\infty} = \frac{1}{a_o + S(-1)} \tag{31}$$

From equation (3)

$$a_kG_k = \frac{1}{1 + \frac{G_{k+1}}{a_k}} > 1$$

Therefore

$$0 > \frac{G_{k+1}}{a_k} > -1$$

Multiplying through by $a_k a_{k+1}$, a negative number,

$$1 < a_{k+1}G_{k+1} < -a_k a_{k+1} = b_k b_{k+1}$$

Since $b_k b_{k+1}$ is greater than unity, so is b_k or b_{k+1} also, and therefore $\Sigma\, b_k$ diverges. $S(-1)$ is then convergent providing $z = -1$ falls on a segment of the negative real axis cut which contains no singularities. Equation (31), with the substitutions indicated and with r_k replacing a_kG_k, can be recast in the form

$$G_{o\infty} = \cfrac{G_o(\frac{1}{r_o})}{1 + \cfrac{(1 - \frac{1}{r_o})\frac{1}{r_1 z}}{1 + \cfrac{(1 - \frac{1}{r_1})\frac{1}{r_2 z}}{1 + \cfrac{(1 - \frac{1}{r_2})\frac{1}{r_3 z}}{1 + \cdots}}}} \tag{32}$$

Wall (23) shows that for $0 < \frac{1}{r_k} < 1$, the expansion (32) converges for $\left|\frac{1}{z}\right| \leqq 1$. Thus the cut in the z plane runs from zero to -1 (exclusive), and elsewhere, including at $z = -1$, the expansion converges and G_o^* exists.

Unfortunately, however, in this case G_o^* does not necessarily equal G_o. The convergent continued fraction can be placed in the form

$$G_o^* = \cfrac{G_o}{r_o + \cfrac{1 - r_o}{r_1 + \cfrac{1 - r_1}{r_2 + \cdots}}}$$

Where the r_k repeat after r_n, the expansion can be terminated as shown below.

$$G_o^* = \cfrac{G_o}{r_o + \cfrac{1 - r_o}{r_1 + \cfrac{1 - r_1}{r_2 + \cdots \cfrac{1 - r_{n-1}}{r_n + (1 - r_n)\frac{G_o^*}{G_o}}}}}$$

By inspection $G_o^* = G_o$ is formally one solution, and the other is found by forming and solving the pertinent quadratic equation. Wall (24) gives the criteria for identifying which solution actually represents the convergence limit; in the case under consideration here it is always the smaller one.

Special cases of interest are:

(a) All r_k are the same and equal r. G_o^* equals the smaller of G_o and $\frac{G_o}{r-1}$. Thus if r is greater than two, the expansion does not converge to G_o.

(b) All r_k, for k even, equal r_o, and for k odd, equal r_1. G_o^* equals the smaller of G_o and $\frac{r_1 G_o}{r_o(r_1-1)}$; the expansion converges to G_o only if $\frac{1}{r_o} + \frac{1}{r_1} \geqq 1$.

A comparison of the two methods of selecting $a_k G_k$ shows, in brief, that where all $a_k G_k$ are less than unity, the expansion may or may not converge, but if convergent, the limit is G_o; and that where all $a_k G_k$ are greater than unity the expansion always converges, but the limit may or may not equal G_o.

<u>Comparison series</u>.--A useful relationship can be obtained by comparing a given expansion with one selected from the cases of special interest mentioned in the preceding section. The series of approximants of the selected expansion is called a comparison series. An inequality relationship between the approximants of the given expansion and the comparison series is developed as follows.

Let the subindex m denote terms of the comparison series. Equation (18) is used to set up the following ratio.

$$\frac{\dfrac{G_{kn}}{G_k}}{\dfrac{{}_mG_{kn}}{{}_mG_k}} = \frac{1 + ({}_ma_k\,{}_mG_k - 1)\left(1 - \dfrac{{}_mG_{k+1,n}}{{}_mG_{k+1}}\right)}{1 + (a_kG_k - 1)\left(1 - \dfrac{G_{k+1,n}}{G_{k+1}}\right)} \tag{33}$$

Inspection of (33) confirms the following inequalities.

(1) $a_k G_k$ and ${}_m a_k \, {}_m G_k$ less than unity:

If ${}_m a_k \, {}_m G_k < a_k G_k$, and if

$$\frac{{}_m G_{k+1,n}}{{}_m G_{k+1}} \gtrless \frac{G_{k+1,n}}{G_{k+1}} \gtrless 1 \tag{34}$$

then

$$\frac{{}_m G_{kn}}{{}_m G_k} \lessgtr \frac{G_{kn}}{G_k} \lessgtr 1 \tag{35}$$

either the upper or lower inequality signs holding throughout. The last part of inequality (35) is obtained from (19).

(2) $a_k G_k$ and ${}_m a_k \, {}_m G_k$ greater than unity:

If ${}_m a_k \, {}_m G_k > a_k G_k$, and if

$$\frac{{}_m G_{k+1,n}}{{}_m G_{k+1}} < \frac{G_{k+1,n}}{G_{k+1}} < 1 \tag{36}$$

then

$$\frac{{}_m G_{kn}}{{}_m G_k} < \frac{G_{kn}}{G_k} < 1 \tag{37}$$

The last part of (37) is obtained from (20).

Now since $\dfrac{\dfrac{G_{nn}}{G_n}}{\dfrac{{}_m G_{nn}}{{}_m G_n}} = \dfrac{{}_m a_n \, {}_m G_n}{a_n G_n}$ is less than one in the first case and greater than one in the second, and since ${}_m G_o = G_o$ (as both expansions are based on the same given function), the various inequalities derived from (33) can be extended through successive stages to yield the following simple results.

(1) If ${}_m a_k \, {}_m G_k < a_k G_k < 1$, then

$$\left(\frac{{}_m G_{on}}{G_o}\right)^{(-1)^n} > \left(\frac{G_{on}}{G_o}\right)^{(-1)^n} > 1 \tag{38}$$

(2) If ${}_m a_k \, {}_m G_k > a_k G_k > 1$, then

$$\frac{{}_m G_{on}}{G_o} < \frac{G_{on}}{G_o} < 1 \tag{39}$$

In words, under the conditions given in (38) and (39), the approximants of the given expansion are better than those of the comparison series.

Any selection of comparison series might be made, but in this paper numerical results will be obtained only for the series for which ${}_m a_k \, {}_m G_k = {}_m r_k = m$, that is, all ${}_m r_k$ have the same value. The series $\frac{{}_m G_{on}}{G_o}$ may be tabulated for various values of m and n, but the results will be in a more useful form if expressed in logarithmic terms. Let

$$M = 10 \log m = {}_m H_k + {}_m A_k = {}_m H_k - (- {}_m A_k), \text{ and}$$

$$D = 10 \log \frac{{}_m G_{on}}{G_o} = {}_m H_{on} - H_o.$$

In Appendix D, values of D are tabulated for various values of M and n. The tables may be used in the following manner. Suppose that in a certain expansion of a given H_o, for which the subscript of the last element is n, it is observed that the difference between every H_k and the corresponding $-A_k$ falls between zero and M db. From the tables a value of D db. is obtained which represents the maximum deviation from H_o that H_{on} can have. If this discrepancy meets the tolerance requirements for an acceptable approximant the expansion may be halted at this point. The inequalities in the analysis may also all be reversed, with the result that if every $H_k + A_k$ is greater in magnitude than M, the last approximant will be worse than that indicated by the value of D obtained from the tables.

Deviations from a consistent rule for the magnitude of a_kG_k.--In preceding sections the development has been based on the assumption that a_kG_k for a particular expansion are either all greater than unity or all less than unity. In this section deviations from this rule will be considered.

(1) Consider first a case in which successive a_kG_k are positive, but may be greater or less than one in some random combination. From equation (17)

$$\frac{dg_k}{dg_{k+1}} = \frac{-1}{(a_k + g_{k+1})^2}$$

$$\frac{dg_o}{dg_n} = (-1)^n \prod_{i=0}^{n-1} \frac{1}{(a_i + g_{i+1})^2} \tag{40}$$

Thus g_o is a monotonically increasing or decreasing function of g_n. Corresponding to the possible values 0, G_n, G_{nn}, and $\pm\infty$ for g_n, g_o takes on respectively the values $G_{o,n-1}$, G_o, G_{on}, and $G_{o,n-2}$. From the positiveness of a_nG_n, G_{nn} and G_n have the same sign, and so both fall between zero and either $+\infty$ or $-\infty$. From (40) it follows that G_o and G_{on} fall between $G_{o,n-1}$ and $G_{o,n-2}$. The term "fall between" embraces the concept that $+\infty$ is connected with $-\infty$, so that if $G_{o,n-1}$ and $G_{o,n-2}$ do <u>not</u> bracket G_o, G_{on} may have any value, including a negative or infinitely large value, that falls outside the bracket. Evidently under these conditions the expansion may or may not converge.

(2) Suppose all a_kG_k are negative. From equation (5) all G_k will be positive, and hence all a_k will be negative. G_{on}, being composed of negative elements, must be negative and cannot converge to G_o.

(3) Let a certain a_p equal zero. Then

$$G_{p+1} = \frac{1}{G_p}$$

$$G_{op} = G_{o,p-2}$$

$$G_{o\infty} = \cfrac{1}{a_o + \cfrac{1}{a_1 + \cdots + \cfrac{1}{a_{p-1} + a_{p+1} + \cfrac{1}{a_{p+2} + \cdots}}}}$$

Note that the disappearance of the a_p term means that a_{p-1} and a_{p+1} are added directly to form a single element. This form of expansion is often convenient and may be obtained merely by inverting G_k at any desired stage before proceeding with the expansion. It does not affect convergence properties.

(4) Consider a case in which a certain a_pG_p equals one. Then of course $G_{op} = G_o$, and if the expansion can be halted at this stage the desired approximation is exactly achieved. But the condition may hold only at an isolated frequency, and the situation at other frequencies may require a continuation of the expansion.

$$G_{p+1} = \frac{1}{G_p}(1 - a_pG_p) = 0$$

It is not practically feasible to make $a_{p+1}G_{p+1} > 0$, particularly when working on logarithmic scales, and therefore:

$$G_{p+2} = \frac{1}{G_{p+1}} - a_{p+1} = \infty$$

$$G_{p+3} = \frac{1}{G_{p+2}} - a_{p+2} = - a_{p+2}$$

The effect of this development on the approximant may be investigated by using the following recurrence formulas (25).

$$\text{Let } G_{ok} = \frac{P_{ok}}{Q_{ok}}. \text{ Then}$$

$$P_{ok} = a_kP_{o,k-1} + P_{o,k-2} \tag{41}$$

$$Q_{ok} = a_kQ_{o,k-1} + Q_{o,k-2} \tag{42}$$

In the case in question

$$G_{op} = \frac{a_p P_{o,p-1} + P_{o,p-2}}{a_p Q_{o,p-1} + Q_{o,p-2}} = \frac{\frac{1}{G_p} P_{o,p-1} + P_{o,p-2}}{\frac{1}{G_p} Q_{o,p-1} + Q_{o,p-2}} = G_o$$

By appropriate substitutions it may be shown that

$$G_{o,p+2} = G_o \left[\frac{1 + \frac{P_{o,p-1}}{P_{op}} \left(\frac{1}{a_{p+1} + \frac{1}{a_{p+2}}} \right)}{1 + \frac{Q_{o,p-1}}{Q_{op}} \left(\frac{1}{a_{p+1} + \frac{1}{a_{p+2}}} \right)} \right]$$

From the foregoing it is evident that if a_{p+1} is large and a_{p+2} is small, and both have the same sign, $G_{o,p+2}$ will be close to G_o. Thus situations of this type can be handled if the expansion must be continued.

(5) In cases (1) and (2) above divergent expansions were shown to be possible results. If, contrary to the hypotheses of those paragraphs, deviations from a consistent $a_k G_k$ magnitude rule occur only for a finite number of stages, say, prior to the p^{th} stage, then a convergent expansion will result providing G_{pk} converges to G_p. This follows from the monotonic variation cited in equation (40). In the methods to be outlined in Chapters V and VI deviations from a consistent convergence-producing rule must be accepted, but the considerations of this section show that a satisfactory approximant may still be obtained.

CHAPTER IV

APPROXIMATION WITH GENERAL RATIONAL FUNCTIONS

In Chapter II several possible methods of expanding a prescribed function were illustrated. The last four of these involved elements asymptotic to the prescribed function or its remainders. Approximation using asymptotic elements will be considered in Chapters V and VI. The other two expansions involved elements that were general rational functions, and illustrate the method to be taken up in this chapter.

Rule for the magnitude of $a_k G_k$.--This method might also be described as the method of uniform approximation throughout the frequency range. By this it is meant that the criterion used in selecting successive $\frac{1}{a_k}$ is that they fall as close to the corresponding G_k as is feasible at all frequencies. This is in contrast with the methods employing asymptotic elements in that they give a preference to the approximation at the frequency of the asymptotic limit.

In accordance with the development of the preceding chapter a consistent rule for the magnitude of $a_k G_k$ should be established; in a particular problem that product should be greater than unity for every stage, or it should be less than unity for every stage. For the bulk of the figures and discussion in this chapter the case in which $a_k G_k$ are greater than unity is portrayed. Basic formulas apply to both cases, and the behavior of successive remainders as influenced by the shape of the element functions is quite similar in both cases. A notable difference is that in the case for which $a_k G_k$ are greater than unity, successive a_k and G_k alternate in sign, whereas for the other case they are all positive. As a result the use of the former case as a frame for the presentation offers the advantage of illustrating the handling of negative elements in the logarithmic approach of the method. At the conclusion of the chapter the significant points wherein the method for $a_k G_k$ less than unity differs from that already presented are set forth.

Procedure and logarithmic portrayal.--Suppose that a prescribed amplitude function of frequency is given. If not already so expressed, it is first converted to logarithmic form, with ordinates in decibels and abscissas in frequency decades. In this form the prescribed function is denoted $H_o(v)$, or, for brevity, just H_o, and is sketched or plotted as shown in Figure 6. The H_o of Figure 6 is a typical function, being finite in the range of interest and having a 'zero' only at an extremity of the frequency range, in this case at infinity. (Of course, it is G_o and not H_o that really has a zero at infinity; H_o has the corresponding logarithmic singularity. However, the terms pole and zero are so much briefer and more descriptive than logarithmic singularity that they will be used to describe the behavior of H_o that is due to their prescence in G_o. No ambiguity results.) If H_o is not already given in tabulated form it should be tabulated in the range of interest. Increments of one-tenth of a decade on the log ω scale usually afford a dense enough sampling. Where H_o changes significantly in a narrow frequency range a smaller interval

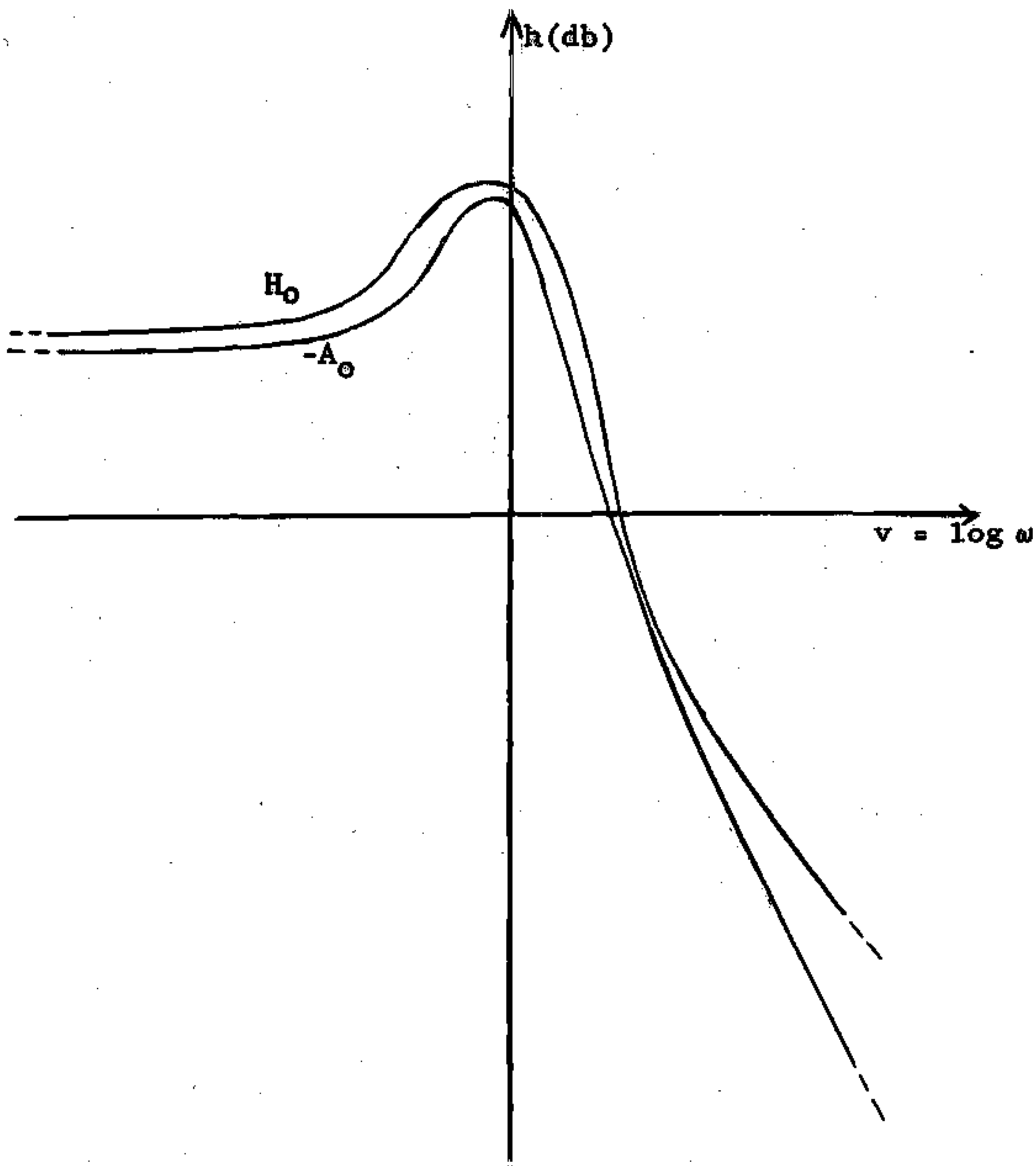

Fig. 6. Typical Prescribed Function and First Element.

between abscissas may be advisable. If it develops in a later stage of the process that smaller intervals are required in some frequency range, they can readily be inserted with a few simple calculations.

The next step is to select A_o. Recalling that the strategy of selection prescribes that $a_k G_k$ be greater than unity, it follows that $A_k + H_k$ must be positive. Accordingly $-A_o$ is chosen to be the logarithmic representation of a rational function, and to approximate H_o but be everywhere less than H_o, as shown in Figure 6. $-A_o$ is formed as the sum of standard components of the type mentioned in Chapter I. Appendix B gives a tabulation for ready reference of standard components of the forms

$$10 \log \left[1 + \left(\frac{\omega}{\omega_k}\right)^2\right] \text{ and}$$

$$10 \log \left[1 + 2c\left(\frac{\omega}{\omega_k}\right)^2 + \left(\frac{\omega}{\omega_k}\right)^4\right]$$

Either form of equation (12) is used in conjunction with the tables of $L(u)$ (given in Appendix C) to calculate H_1 point by point. H_1 is sketched in Figure 7. Since $A_o + H_o$ is positive $L(-A_o - H_o)$ is negative. From equation (12) the real part of H_1, the part that is plotted, will therefore fall below A_o as shown in Figure 7. The previous statement referred to the real part of H_1 because, of course, G_1 is negative, as are all G_k for k odd in this method. Thus $H_1 = \mathrm{Re}(H_1) + \mathrm{I} = (H_1 + \mathrm{I}) + \mathrm{I}$. It is important to carry the symbol I on the graphs and in the tabulated values where appropriate. In subsequent chapters, where instances are considered in which $-A_k$ may cross H_k, it will be found that G_{k+1} changes

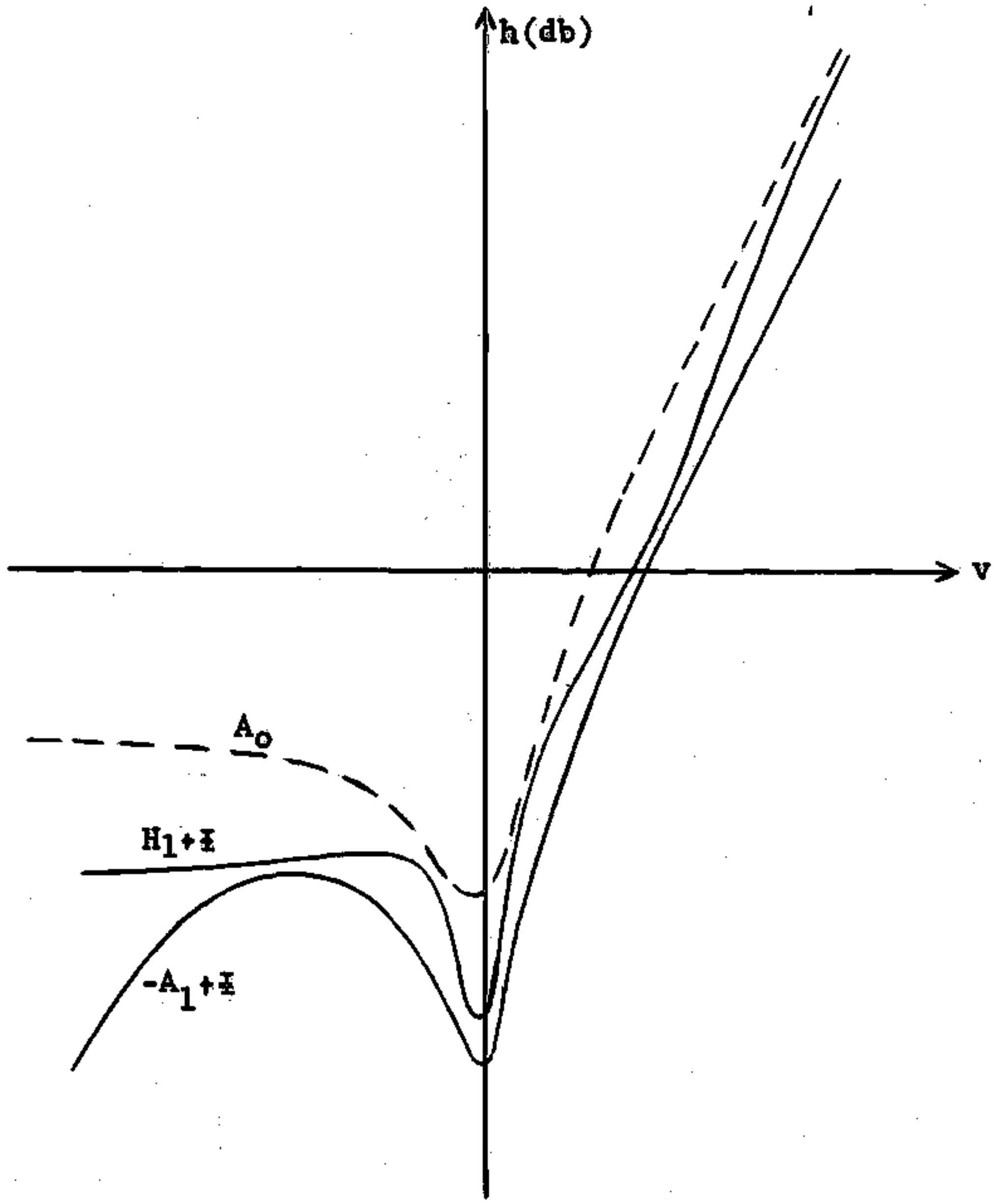

Fig. 7. Remainder and Succeeding Element.

sign, and that therefore some segments of H_{k+1} will carry the $\mathtt{I}$ symbol and others will not.

The process is now repeated and a $-A_1$ is chosen to approximate H_1. Since H_1 is complex, so is $-A_1$, its real part being $-A_1 + \mathtt{I}$ as shown plotted in Figure 7.

Two approximants to H_o have now been obtained. They are:

$$H_{oo} = 10 \log G_{oo} = 10 \log \frac{1}{a_o} = -A_o$$

$$H_{o1} = 10 \log \left(\frac{1}{a_o + \frac{1}{a_1}}\right) = -A_o - L(-A_o + \mathtt{I} - A_1)$$

The latter may be computed point by point. Note from Figure 7 that $-A_1 + \mathtt{I} - A_o$ is negative and hence so is $L(-A_o + \mathtt{I} - A_1)$. It follows that H_{o1} falls above H_{oo}. All the pertinent curves are replotted together in Figure 8.

<u>Shape of approximants and remainders</u>.--It is by now evident that a fair amount of art is involved in the selection of successive A_k. A study of Figure 8 reveals some of the properties of the curves which are useful in contributing to this art.

In the first place, H_{o1} falls above H_{oo} and below H_o. If H_{oo} is an acceptable approximant to H_o, then H_{o1} is bound to be acceptable also in the same frequency range, as, for example, to the left of point a in the figure. Therefore the behavior of $-A_1 + \mathtt{I}$ in this range is unimportant as long as it falls below $H_1 + \mathtt{I}$. This property derives from inequality (23)

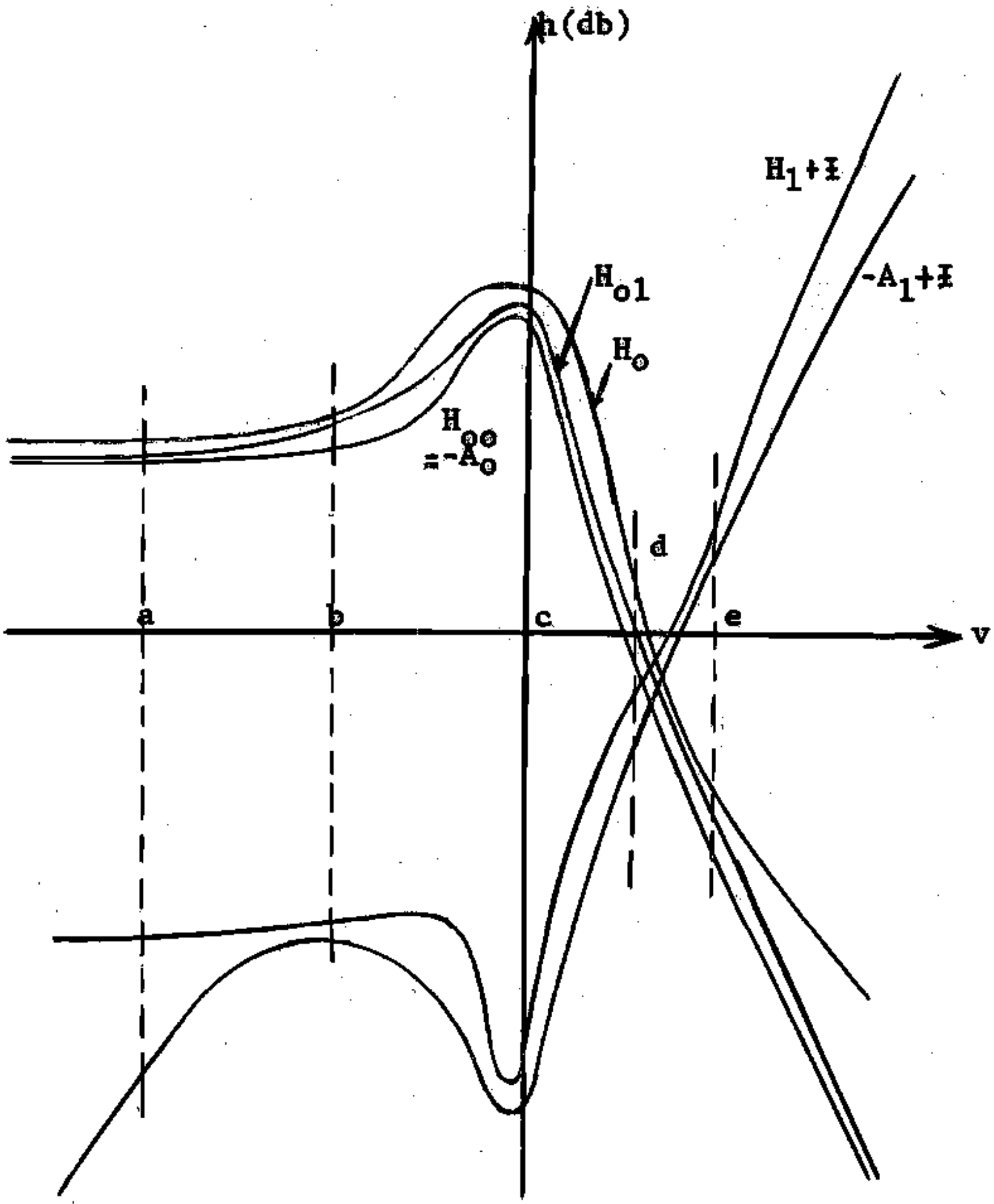

Fig. 8. Shape of Approximants and Remainder.

developed in the preceding chapter.

Secondly, H_{o1} tends toward H_o rather than H_{oo} at points where $-A_1 + I$ is closer to $H_1 + I$, as at point b in Figure 8. If $-A_o$ ($= H_{oo}$) is less than three decibels below H_o at this point, $H_o - H_{o1}$ is less than $H_1 + I - (-A_1 + I)$. If the latter is less than the permissible tolerance prescribed for the desired approximant, then H_{o1} is acceptable at that frequency. This property derives from inequality (26), and may be combined with (23) to yield the fact that all higher order approximants will also be acceptable at the same frequency.

A third relation which may be useful is illustrated by the behavior of the curves between points d and e. In that region $A_o + H_o$ and $A_1 + H_1$ are both less than two decibels. From relationship (39) of the preceding chapter, in conjunction with the tables of Appendix D, H_{o1} is less than 0.85 db. below H_o.

Another characteristic illustrated in Figure 8 is the sharp dip in $H_1 + I$ in the vicinity of point c. This occurs partly because $-A_o$ has a rise at that point (recall that $H_1 + I$ must lie below A_o), but mostly because $-A_o$ is close to H_o in a relatively narrow range in the vicinity of c. This sharp dip in $H_1 + I$ makes selection of a suitable $-A_1 + I$ more difficult. It suggests that a better choise for $-A_o$ would be a function that did not peak quite as sharply at c, if one could be found.

<u>Element selection at ends of the frequency range</u>.--A behavior related to the dip in $H_1 + I$ discussed above may arise at the ends of the frequency range of interest, and is illustrated in Figure 9 by the shape of $H_1 + I$

at the low frequency end. Because $-A_o$ is so close to H_o, $H_1 + \text{Ɨ}$ dips very sharply there. This makes the selection of $-A_1 + \text{Ɨ}$ difficult in that region and requires it to be of an unnecessarily high order to remain below $H_1 + \text{Ɨ}$. If $-A_o$ can be placed instead at the lower limit of tolerance permitted for the approximation to H_o, as indicated in Figure 10, a less restrictive $H_1 + \text{Ɨ}$ is obtained while the final result is certain to meet the prescribed tolerance.

<u>Remainder with constant slope</u>.--As pointed out above, it is difficult to find an element to approximate closely a remainder which contains large narrow excursions from its general shape. The easiest remainder to approximate is one having a constant slope which is an integral multiple of twenty decibels per decade. Accordingly, it is frequently desirable to choose an element in such a way as to produce this linear behavior in the succeeding remainder over at least a part of the frequency spectrum.

To investigate how such a choice might be made let

$$H_1 + \text{Ɨ} = 20pv + K,$$

in which p is an integer (positive, negative, or zero) and K is an arbitrary constant. From equation (8)

$$L(A_o + H_o) = H_o + H_1 = H_o + (H_1 + \text{Ɨ}) + \text{Ɨ}$$

$$= H_o + \text{Ɨ} + 20pv + K$$

Applying equation (10)

$$A_o + H_o = L(H_o + \text{Ɨ} + 20pv + K)$$

$$A_o = - H_o + L(H_o + \text{Ɨ} + 20pv + K)$$

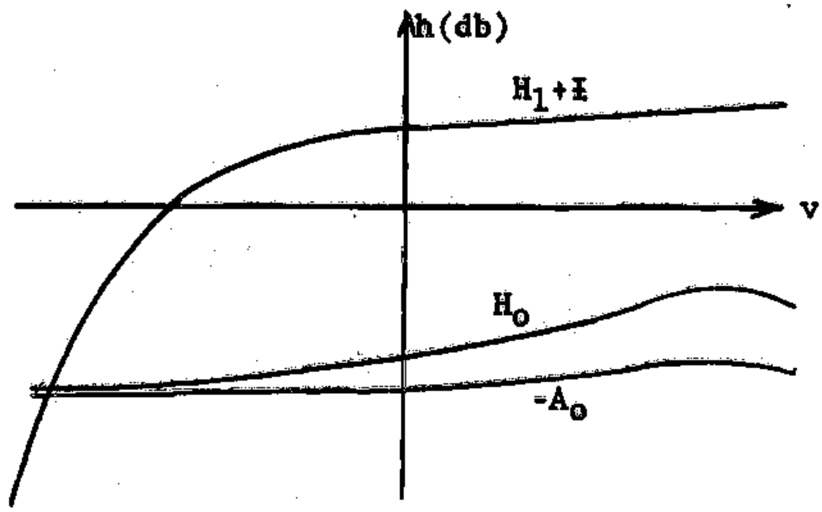

Fig. 9. Undesirable Depression of Remainder at Low Frequencies.

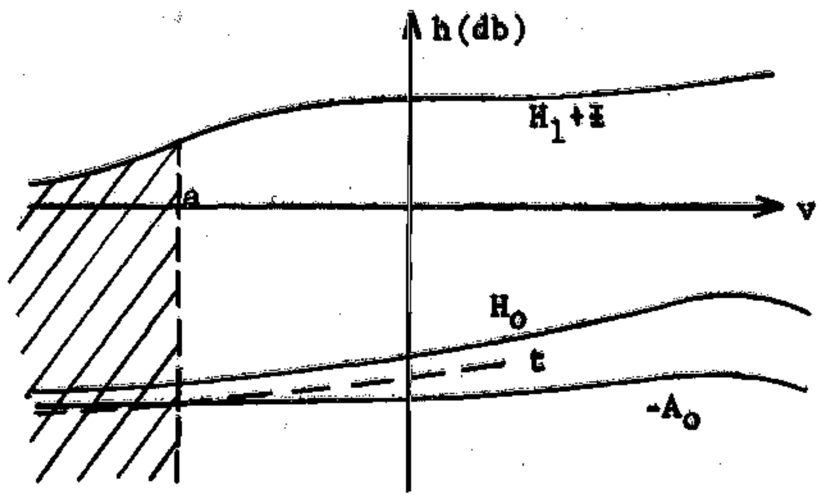

Fig. 10. Preferable Shape of Remainder.

The dotted line labeled "t" in Figure 10 is the lower tolerance limit for the desired approximation to H_o. To the left of point a, the next element, $-A_1+\bar{I}$, may be permitted to fall anywhere in the shaded region.

In more general form, for k even or odd,

$$A_k = - H_k + L(\mathrm{Re}(H_k) + Ɨ + 20pv + K) \qquad (43)$$

will produce a linear $\mathrm{Re}(H_{k+1})$.

Figure 11 illustrates the manner in which equation (43) is applied. The region in which it is most desirable to produce a linear remainder is at frequencies adjacent to the ends of the spectrum of prescribed behavior, and the example of the figure is chosen at the low frequency end. H_o is given, and a $-A_o$ is sought that approaches H_o in such a manner as v decreases that $H_1 + Ɨ$ has a 20p decibel per decade slope. The curves shown in Figure 11 represent possible linear $H_1 + Ɨ$ for different choices of p and K in equation (43). The equation is used to determine the required $-A_o$ corresponding to each; an attempt can then be made to fit one of the standard component curves (or a simple combination of them) to one of the suitable $-A_o$ curves. Alternatively, the normalized function for $-A_o$ may be known, but its exact location is sought. Should it be moved a little to the left or right, or up or down? In this case the normalized $-A_o$ is shifted around until it offers a good fit to one of the $-A_o$ shown in Figure 11.

In the algebraic domain what we have done in the above procedure is to confine attention far enough to the left on the frequency scale that

$$G_o \approx \frac{1}{a_o(\omega^2) + (-b\,\omega^{2p})} \quad \text{(for } \omega \text{ less than some } \omega_m\text{)},$$

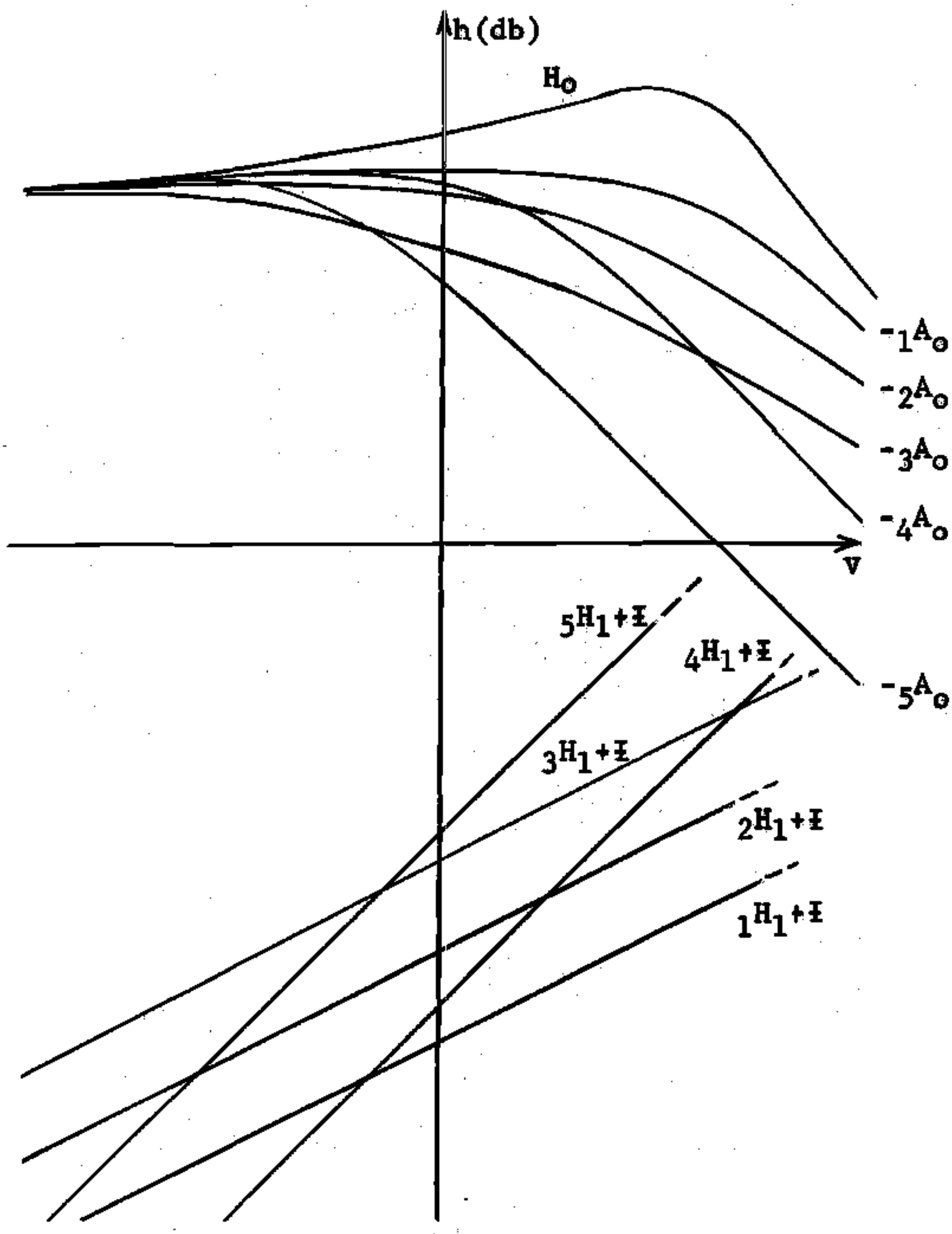

Fig. 11. Production of Linear Remainder.

Subindices identify each remainder with the corresponding $-A_o$ that will produce it.

in which a_o is a recognizable function. By 'recognizable' it is meant that a_o is composed of only a few of the standard components. But if the above relation holds, then certainly

$$G_o \approx \frac{1}{a_o(\omega^2) + c\,\omega^{2p} - (b + c)\omega^{2p}} = \frac{1}{a_o'(\omega^2) - b'\omega^{2p}}$$

also hold for the same frequencies, c being an arbitrary constant. We should expect to find a function a_o appropriate to any value of b in the linear remainder $-b\,\omega^{2p}$ (providing b is sufficiently large). This is true; the difficulty comes in recognizing $a_o(\omega^2)$. For example, if

$$G_o \approx \frac{1}{\dfrac{1 + \omega^2}{1 + 2d\,\omega^2 + \omega^4} - b\omega^2} \qquad \text{for } \omega \text{ less than } \omega_m,$$

then

$$G_o \approx \frac{1}{\dfrac{1 + (c + 1)\,\omega^2 + 2cd\,\omega^4 + c\,\omega^6}{1 + 2d\,\omega^2 + \omega^4} - (b + c)\omega^2} \qquad (\omega < \omega_m)$$

In the second case $a_o(\omega^2)$ would certainly be harder to recognize than in the first case. Of course if, in the second case, we considered only the region to the left of ω_q, where ω_q^2 is very much less than $|2d|$, we would have

$$G_o \approx \frac{1}{\dfrac{1 + (c + 1)\,\omega^2 + 2cd\,\omega^4}{1 + 2d\,\omega^2} - (b + c)\omega^2}$$

In this instance a_o is of the same order as in the original example, and would be as easily recognizable, but we would now be successful in producing a linear remainder over a much smaller segment only of the whole frequency spectrum.

Incorporation of prescribed tolerances into the method.--A restrictive factor in applying the semi-graphical approximation method of this chapter is that the strategy adopted for the magnitude of $a_k G_k$ dictates that H_{on} be less than H_o. Thus an approximant is sought that falls between the given function and the lower limit of prescribed tolerance. This is unnecessarily restrictive, because an approximant that falls between the given function and the upper limit of prescribed tolerance would be equally acceptable. The restriction is easily removed by substituting for H_o in the procedure the upper tolerance bound, which is designated H_o^+. Similarly the lower bound is designated H_o^- on logarithmic graphs, while G_o^+ and G_o^- identify the bounds on algebraic scales. The tolerance bounds are carried forward from stage to stage by means of equation (17) in the same manner as the prescribed function. Thus

$$G_k^- = \frac{1}{a_k + G_{k+1}^+} \quad \text{and} \quad G_k^+ = \frac{1}{a_k + G_{k+1}^-} \tag{44}$$

and in logarithmic values,

$$H_{k+1}^{\pm} = \bar{I} + A_k + L(-A_k - H_k^{\mp}) \tag{45}$$

The effect on the procedure of incorporating the tolerance bounds is best illustrated in some simple figures.

In Figure 12 the curve represents equation (17) for k equal to zero. The figure illustrates how the tolerance bounds are projected forward from the g_o stage to the g_1 stage. If in the succeeding stage of the expansion $1/a_1$ can be chosen to fall between G_1^+ and G_1^-, the approximant G_{o1} will satisfy the prescribed tolerances, as it must fall between G_o^+ and G_o^-. The same relationship will hold for G_{ok} with respect to the selection of $1/a_k$ by reason of the monotonic variation of g_o as a function of g_k shown in equation (40). The relationships of Figure 12 are depicted in the logarithmic realm in Figure 13. It is not even necessary to carry the G_k forward from stage to stage, merely the upper and lower tolerance bounds.

Figure 14 illustrates what happens when $-A_o$ falls above H_o^-, that is, when $a_o G_o^-$ is less than unity. G_1^- is negative, as in the example of Figure 12, but G_1^+ is positive. The figure also indicates that for the particular frequency portrayed, G_{oo} is a satisfactory approximant. However, as this may not hold for all frequencies, it may be necessary to continue the expansion and select an element a_1. Evidently $1/a_1$ should fall between G_1^+ and G_1^- if G_{o1} is to be a satisfactory approximant. On logarithmic scales this situation might appear as shown by the curves of Figure 15. Between v = a and v = b H_1^+ does not carry the $\mathbb{I}$ symbol; in other words, G_1^+ is positive in this range, which corresponds to the situation depicted in Figure 14. As far as the selection of $-A_1 + \mathbb{I}$ is concerned, it follows the strategy previously used; we attempt to choose $-A_1 + \mathbb{I}$ to fall between $H_1^- + \mathbb{I}$ and $H_1^+ + \mathbb{I}$. However an interpretation is needed for the middle range between a and b, and it is this: $-A_1 + \mathbb{I}$

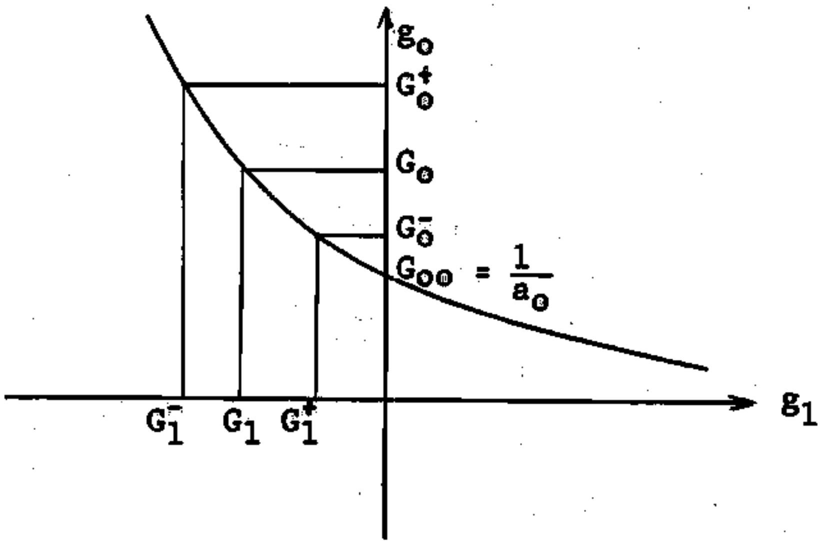

Fig. 12. Projection of Tolerance Bounds.

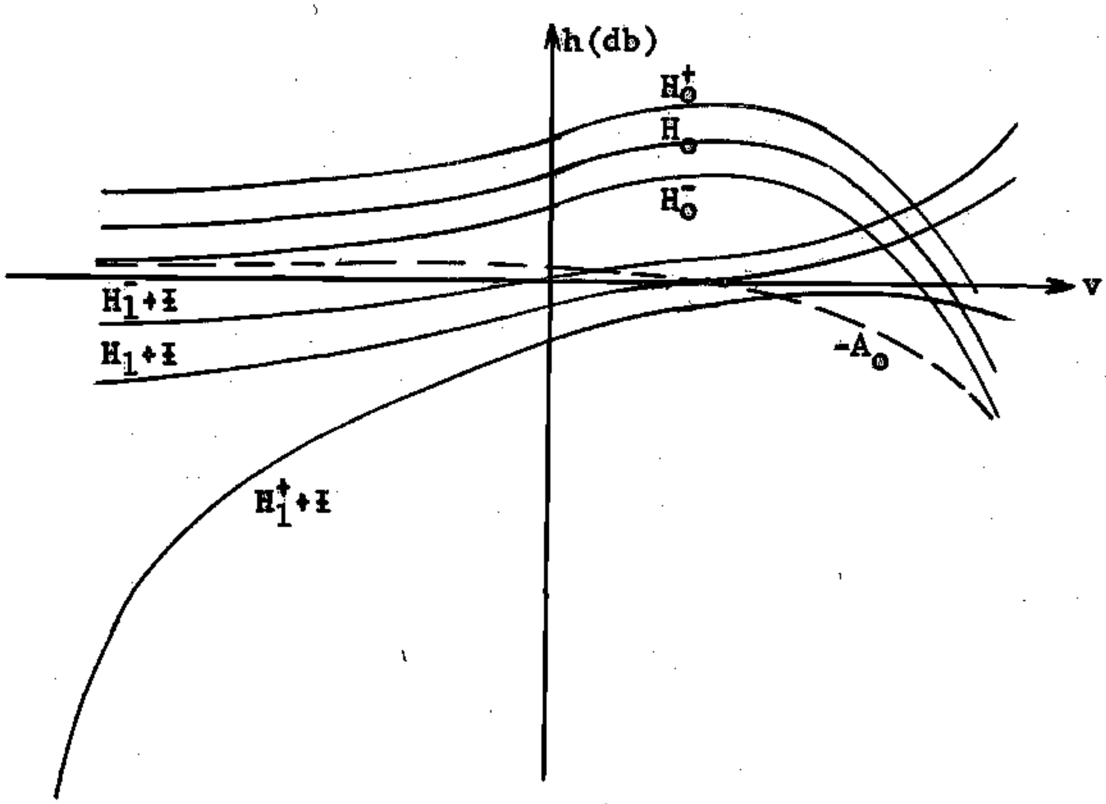

Fig. 13. Tolerance Bounds on Logarithmic Graph.

An ideal $-A_1 + I$ would fall between $H_1^+ + I$ and $H_1^- + I$.

ought to fall anywhere below H_1^- + I down to $-\infty$ + I and back up on the real sheet ($-A_1$ being real instead of complex) from $-\infty$ as far as H_1^+. The word sheet is used as a convenient term to distinguish between the complex range and the real range of logarithmic values. Thus in Figure 15 H_1^+ falls partly on the complex sheet (where it carries the I symbol) and partly on the real sheet. Of course, the real sheet corresponds to the positive axis and the complex sheet to the negative axis of algebraic scales of figures like Figure 14.

Although theoretically $-A_1$ may thus be permitted to fall partly on the real sheet and partly on the complex sheet, it must actually fall entirely on one sheet or the other because it is to be composed of standard components. The latter, as tabulated in Appendix B, are all positive throughout the real frequency range, but may be preceded by a negative sign for algebraic values and thus be complex at all frequencies on logarithmic scales. Factors of the form $(\omega^2 - \omega_k^2)$ have not been included in the standard components; theoretically they could be, but practically they are not found to be necessary and their omission contributes to simplicity. Reviewing Figure 15, the location of the tolerance bounds outside the frequency band from a to b indicates that $-A_1$ should be complex. Accordingly, the real portion of H_1^+ may be omitted. Figure 16 illustrates the desirable region for the location of $-A_1$ with this omission incorporated. The omission amounts to replacing the real part of H_1^+ between a and b with the point at $-\infty$ + I.

Partly to illustrate the complications attendant upon carrying the full H_1^+ boundary forward, but also to contribute further to understanding

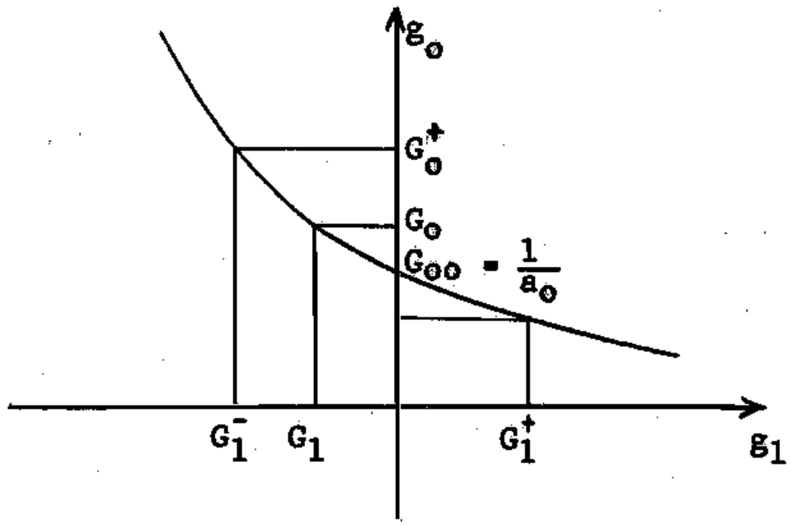

Fig. 14. Remainder Tolerance Bounds with Opposite Signs.

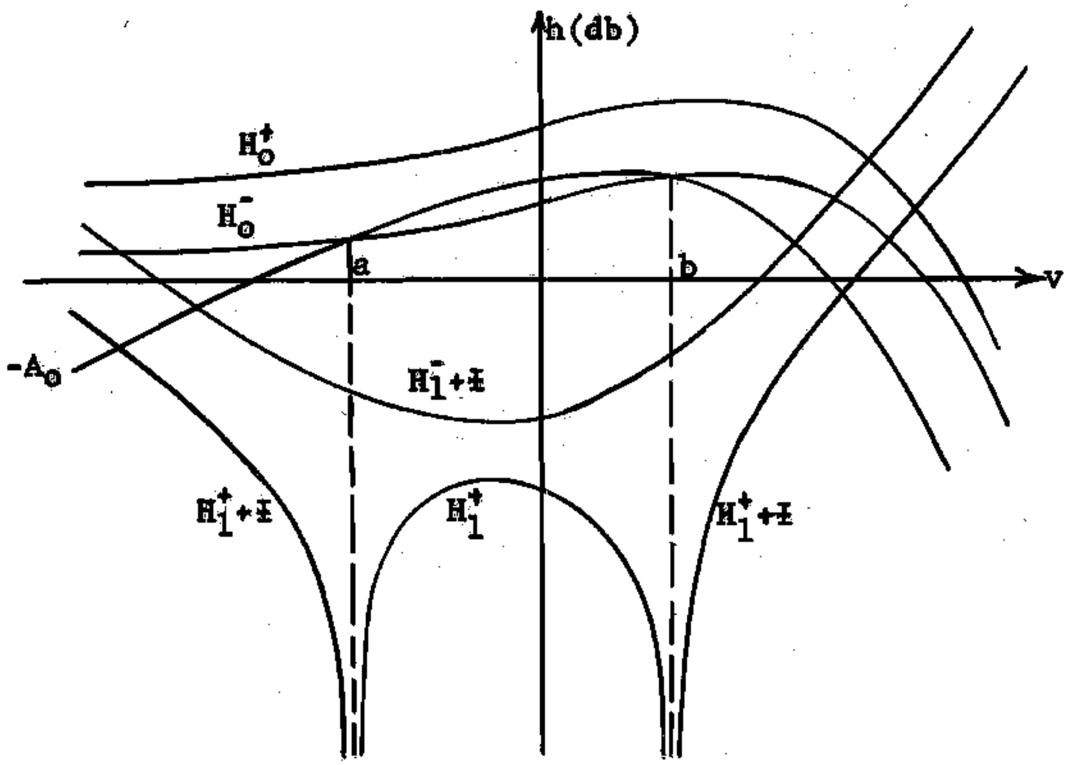

Fig. 15. Tolerance Bound Appearing on both Sheets.

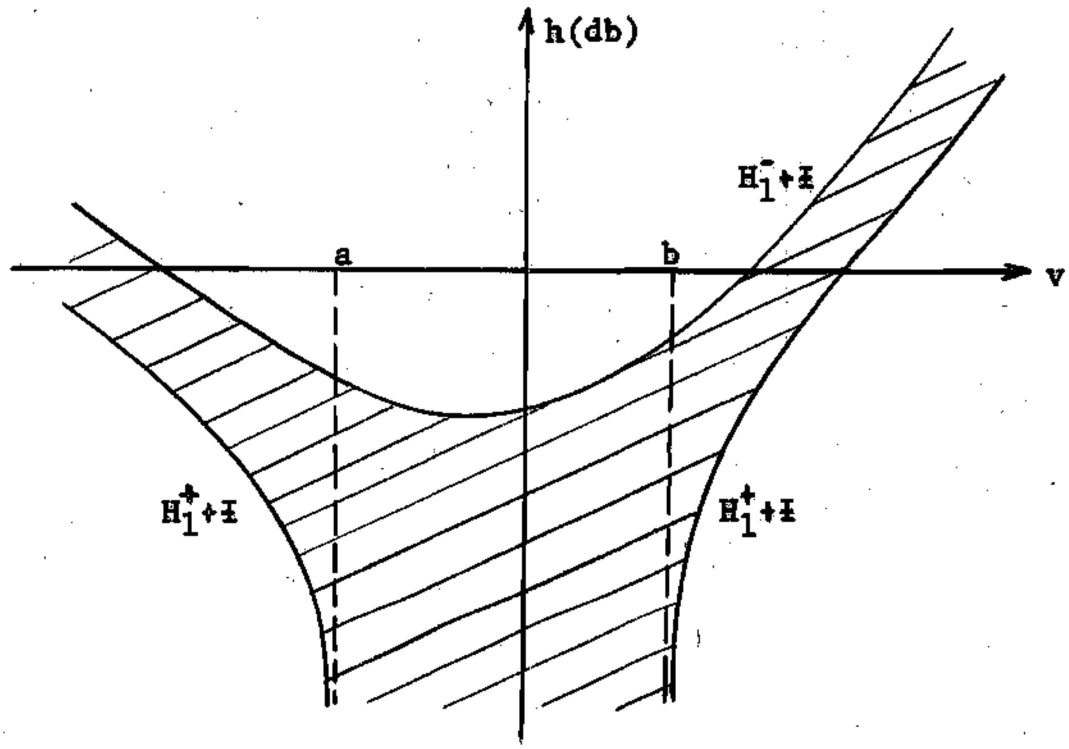

Fig. 16. Pertinent Tolerance Bounds.

The shaded area is the region in which ideally $-A_1 + \bar{I}$ should fall.

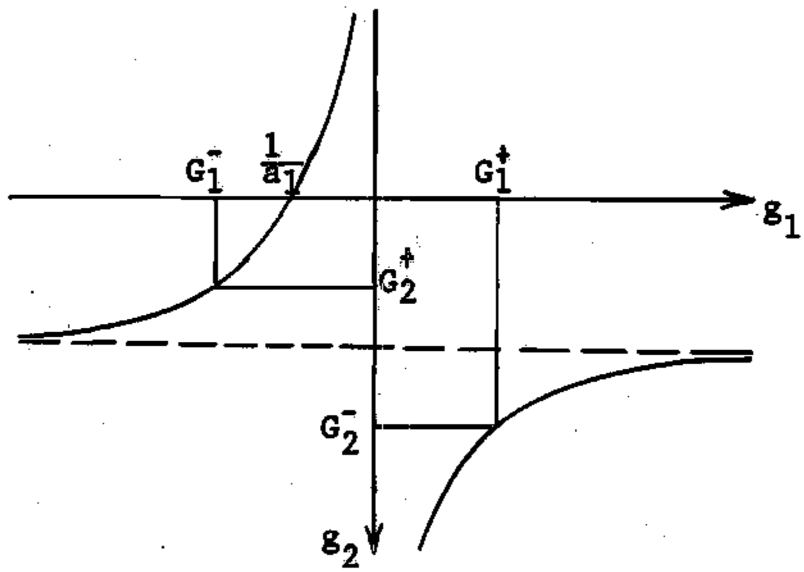

Fig. 17. Further Projection of Tolerance Bounds.

the behavior of the logarithmic curves by investigating some peculiar forms they may assume, consider in the next two figures what happens when the problem of Figures 14 and 15 is carried forward another stage. Figure 17 shows a plot of g_1 against g_2 as an extension of the situation of Figure 14. At first glance it might appear erroneous that G_2^+ should be smaller in magnitude than G_2^-. But further examination shows that the desirable region for $1/a_2$ extends from G_2^+ to zero (upwards on the figure) and on up to $-\infty$ (at the top of the figure), and then picks up again at $g_2 = +\infty$ (bottom of figure) and comes up to G_2^-. Therefore with acceptance of the concept that plus infinity is somehow connected with minus infinity the superscripts correctly identify the upper and lower tolerance bounds. The situation is portrayed to logarithmic scales in Figure 18.

The interpretation of Figure 18, as to what is a desirable $-A_2$, follows:

(1) To the left of a, and between b and c, $-A_2$ would fall anywhere from H_2^+ down to $-\infty$ and back up on the complex sheet to $H_2^- + I$.

(2) To the right of c, $-A_2$ would fall between H_2^+ and H_2^-.

(3) Between a and b, $-A_2$ would fall anywhere from H_2^+ down to $-\infty$, back up on the complex sheet from $-\infty + I$ to $+\infty + I$, and down from $+\infty$ on the real sheet to H_2^-. It may not cross from below H_2^+ to above H_2^- directly on the real sheet without producing a non-realizable G_{o2}, that is, a G_{o2} which is negative for some values of ω.

If segments of the tolerance bounds are omitted at each stage when they are not located on the sheet on which the succeeding element must fall, the much simpler bounds obtained for the second remainder are shown in Figure 19. A desirable $-A_2$ should fall between H_2^+ and H_2^-, or between H_2^+ and $-\infty$, as appropriate.

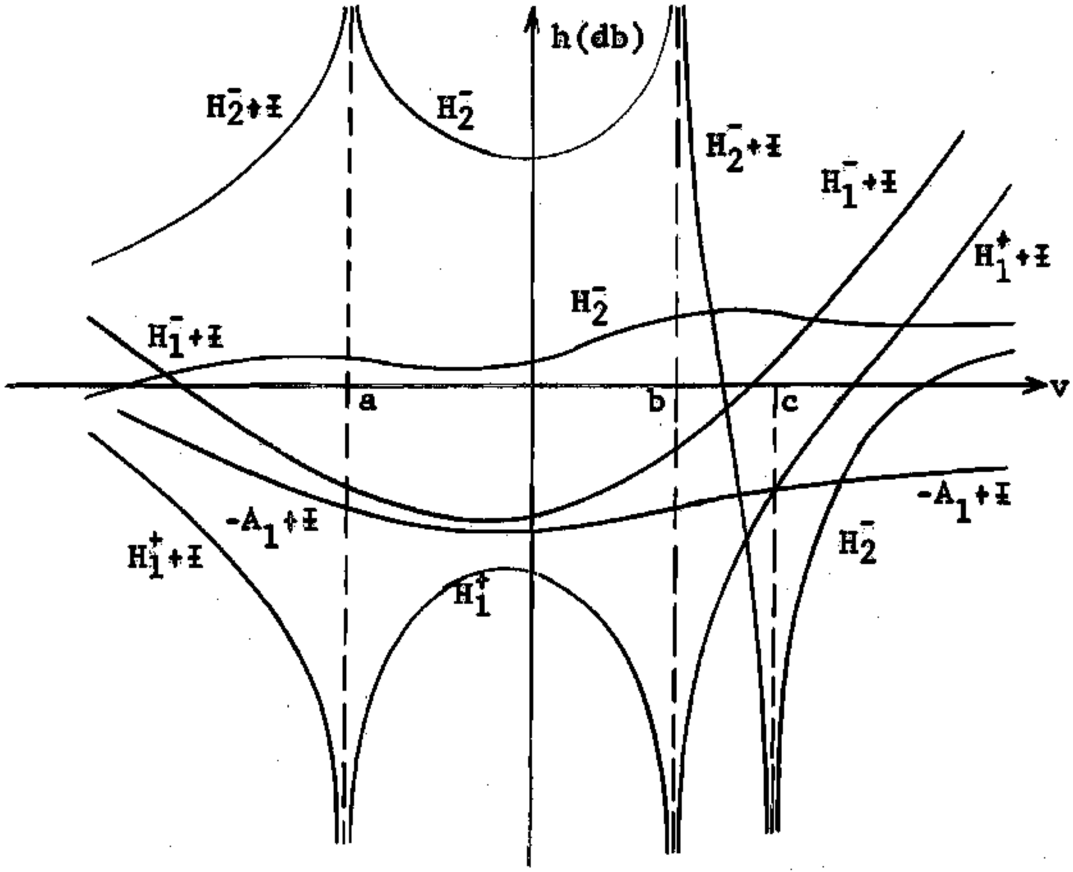

Fig. 18. Complete Tolerance Bounds for Second Remainder.

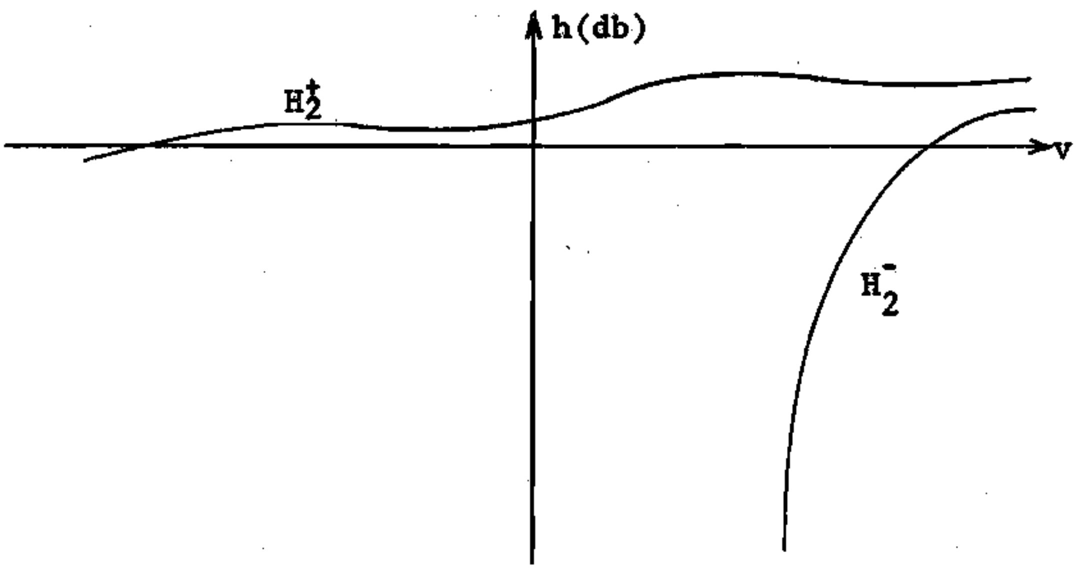

Fig. 19. Pertinent Tolerance Bounds for Second Remainder.

The procedure for incorporating the tolerance bounds into the semi-graphical approximation method has been given above. The advantage of doing so include the ability of the method to handle non-uniform prescribed tolerances and ready determination of the stage when an approximant acceptable throughout the frequency spectrum has been achieved.

<u>Summary of the strategy</u>.--The points previously developed which contribute to the strategy for selecting successive elements are restated below. The statements are made for the case when k is even. For k odd, $-A_k + I$ must be substituted for $-A_k$, $H_k^- + I$ must be substituted for H_k^+, $H_k^+ + I$ must be substituted for H_k^-, and vice-versa.

(1) $-A_k$ must fall below H_k^+. This insures that all G_{on} are less than G_o^+, and that a convergence-producing strategy is being applied. $-A_k$ fall on the same sheet as H_k^+. $-A_k$ are chosen from the table of standard components, or may be the sum of several such components.

(2) $-A_k$ are chosen greater than H_k^- if possible. In regions where this choice can be made, G_{ok} is greater than G_o^-, and this, coupled with (1), means that G_{ok} is an acceptable approximant.

(3) Subject to (2), $-A_k$ is chosen to fall closer to H_k^- rather than H_k^+ in order to produce less violent fluctuations in H_{k+1}^-. This should help to make the choice of the succeeding element easier and of lower order.

In lieu of (2), the lower tolerance bound may be omitted from the calculations and dependence placed instead on equations (23), (26), and (39) in conjunction with Appendix D, checked by direct calculation of apparently favorable approximants, to determine when a satisfactory approximant is achieved.

Examples 2 and 3 in Appendix A illustrate the application of this method to the solution of approximation problems.

<u>Comparison with the method for a_kG_k less than unity.</u>--The exposition of the preceding several sections was all based on the magnitude rule that a_kG_k be greater than unity; or upon the modification, $a_kG_k^{\mp}$ greater than unity (the superscript depending on whether k is even or odd), to include the effect of prescribed tolerances. In this section the differences between that method and the case in which a_kG_k are less than unity are briefly surveyed.

If successive elements are chosen so that a_kG_k are consistently less than unity, $A_k + H_k$ is negative, so that $-A_k$ falls above H_k instead of below it as in the previous case. Equation (12), which applies to both cases, shows that all H_k, and hence all A_k, are real, and that H_{k+1} falls below $-H_k$. The first important difference is that successive approximants do not all improve as k increases. Successive even approximants improve and successive odd approximants improve; thus H_{o2} falls between H_{oo} and H_o, and H_{o3} falls between H_{o1} and H_o; but there is no simple relationship between even and odd approximants. Thus if H_{oo} is a satisfactory approximant in some range, $-A_1$ and $-A_2$ may have any values greater than their corresponding remainders in this range and H_{o2} will be a satisfactory approximant; but there is no guarantee that H_{o1} is any good.

Some miscellaneous properties that are very similar to those of the case when a_kG_k are greater than unity include the following.

(1) $\left|H_{ok} - H_o\right|$ is less than $\left|H_k + A_k\right|$. No three-decibel limit on preceding $\left|H_k + A_k\right|$ is required for this property to hold as it was for the case, a_kG_k greater than unity.

(2) The relationship to comparison series is similar, as discussed in Chapter III.

(3) When $-A_k$ is very close to H_k there is a similar sharp dip in the remainder.

(4) Behavior at ends of the prescribed frequency range is similar, making it desirable to choose $-A_o$ close to the upper limit of the prescribed tolerance.

(5) The equation used to show what choices for $-A_o$ will produce linear remainders is similar but with the $\mp$ symbol omitted. It is

$$A_k = - H_k + L(H_k + 20pv + K) \tag{46}$$

When the method is modified to incorporate upper and lower tolerance bounds, the basic inequality rule becomes:

$$a_k G_k^- < 1$$

This applies whether k is even or odd. In a manner similar to the strategy summary of the preceding section, the strategy in this case may be briefly stated as follows.

(1) $-A_k$ must be greater than H_k^-.

(2) $-A_k$ is preferably less than H_k^+.

(3) $-A_k$ is chosen preferably closer to H_k^+ than to H_k^- in order to reduce fluctuations of the remainder.

A series of figures serves well to illustrate typical behavior of curves resulting from the application of the above strategy. Figure 20 shows a prescribed function, first element, and resulting tolerance bounds of the remainder. Figure 21 shows the shapes of curves obtained when the procedure is extended another stage. Note that in the middle of the spectrum both tolerance bounds go off the real sheet, which is a difference between the behavior of these curves and those of the method illustrated in Figures 15 and 18. Figure 22 portrays the algebraic values of the quantities concerned at a frequency in the middle of the range. Note that for any positive choice for g_2, g_o will fall within the tolerance bounds.

This concludes the exposition of both variations of the first method of semi-graphical approximation presented in this thesis. It is a method by which approximation is achieved, or at least attempted, throughout the frequency range of interest at each stage, and it is a method employing general rational functions as elements. Its chief advantage over the methods of succeeding chapters is that, with fortunate choices for $a_k(\omega^2)$, fewer stages are necessary to reach the final result. It requires considerable art, however, in the selection of successive elements.

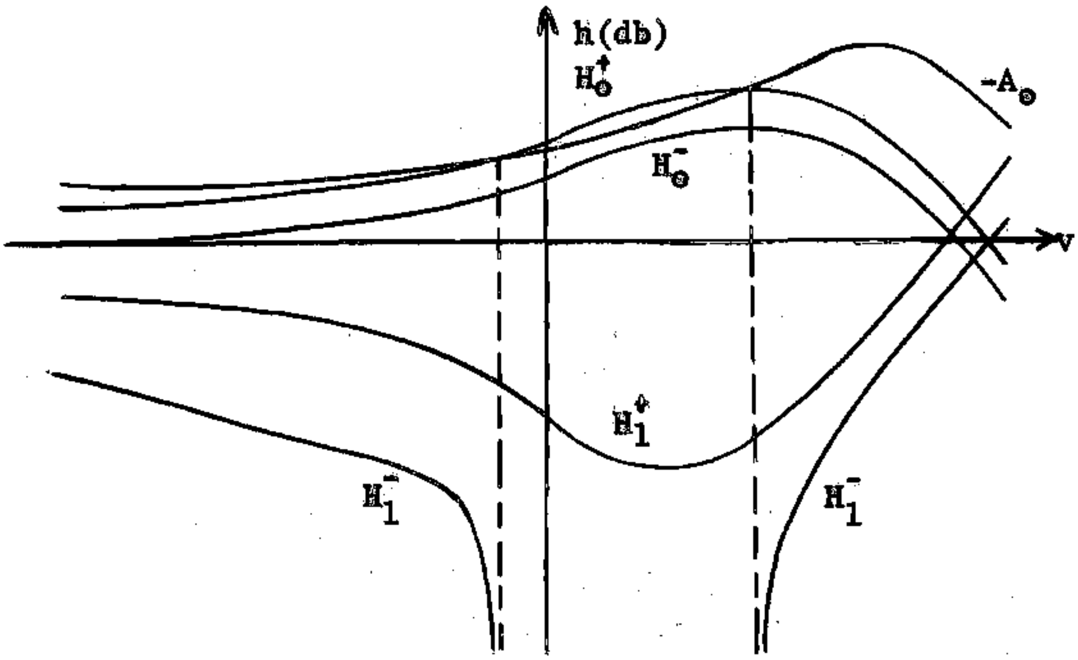

Fig. 20. Curves for the Case $a_k G_k^-$ Less than Unity.

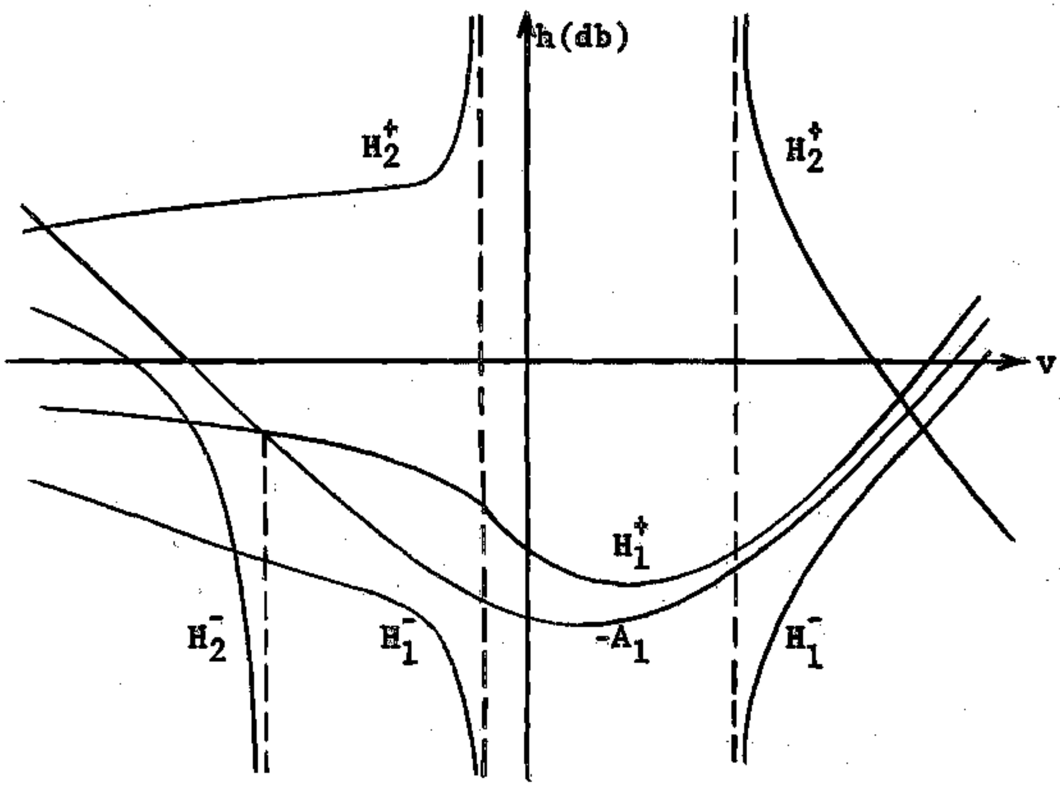

Fig. 21. Second Stage Curves for $a_k G_k^-$ Less than Unity.

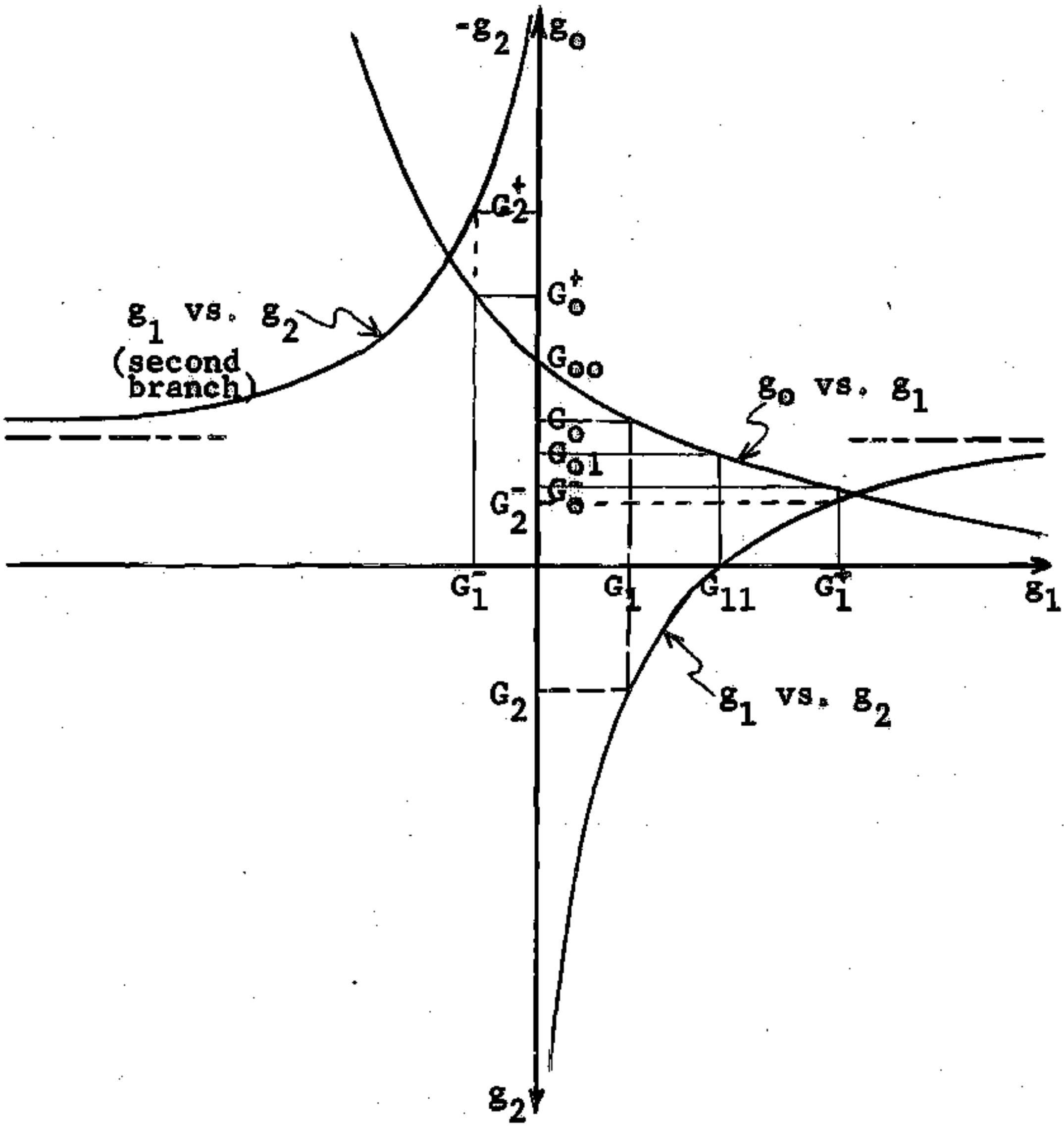

Fig. 22. Projection of Quantities for the Case $a_k G_k^-$ Less than Unity.

In the figure above the vertical axis is both the g_o axis, directed upwards, and the g_2 axis, directed downwards. The figure shows that for any positive choice for G_{22} $(1/a_2)$, G_{o2} will fall between G_{oo} and G_{o1}, and hence between G_o^+ and G_o^-.

CHAPTER V

APPROXIMATION WITH LINEAR FUNCTIONS ASYMPTOTIC AT INFINITY

Restriction on the form of elements.--A weakness of the method set forth in the preceding chapter is the degree of art required in the selection of successive $a_k(\omega^2)$, which are rational functions either positive for all frequencies or negative for all frequencies. Only relatively simple samples of such functions can be readily recognized, although all are composed of standard components of the types tabulated in Appendix B.

In Chapter II several alternative methods of expanding $G_o(\omega^2)$ in continued fraction form were illustrated, among them expansion in terms of monomial functions of ω^2 asymptotic to corresponding remainders as ω increases without bound. The expansion has the form

$$G_{o\infty} = \cfrac{1}{a_o + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}}$$

where $a_k = b_k \omega^{2p_k}$ and $\lim_{\omega \to \infty} a_k G_k = 1$ (47)

The b_k are constants, positive or negative, and p_k are integers, which may be positive, negative, or zero. A_k, the logarithmic representation of a_k against the variable v, is linear, from which derives the title of this chapter. The chapter develops the procedure for using the form (47) of continued fraction expansion as a semi-graphical method of approximation.

<u>Nature of prescribed tolerances at ends of the frequency range of interest.</u>--The first problem that arises is a relatively minor one. What choice should be made for a_o if the asymptotic behavior of H_o is K - 20nv, with n not an integer? Something must be specified about acceptable tolerances before a decision can be made. If the asymptotic behavior of H_o is specified as above, together with a fixed tolerance limit of $\pm N$ decibels, there exists no rational function which can meet the requirements. There must be, for ω greater than some ω_k, a relaxation of the tolerance requirement such that it has the form $\pm(N + 20d \log \frac{\omega}{\omega_k})$, with d large enough so that an integer falls between n + d and n - d, in order for a rational function to exist satisfying the problem requirements. The same is true at the low frequency end of the scale. Often at the extremities of the frequency scale G_o is specified to fall between some constant values; for example, H_o might be specified to be less than -30 decibels at high frequencies, as indicated in Figure 23. In such a case any $-A_o$ with a 20p decibels per decade slope (p being an integer) that falls in the shaded area of the figure to the right of log ω_k is acceptable as the first element.

<u>Intersection of asymptote with prescribed function.</u>--Figure 24 illustrates another situation that will arise in this method. Both ${}_1H_o$ and ${}_2H_o$ have the same asymptote, $-A_o$. In one case $-A_o$ lies entirely above the given function. In the other it lies partly above and partly below. Evidently

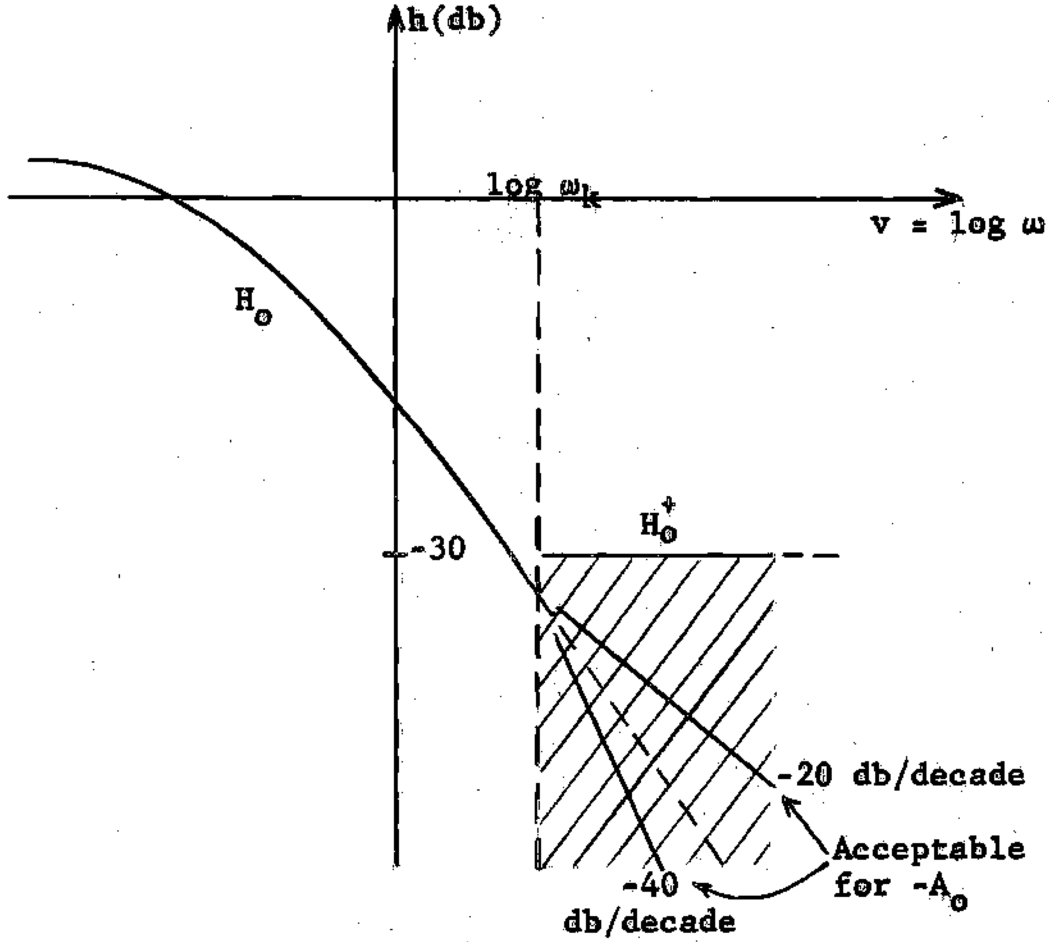

Fig. 23. Typical Tolerance Prescription for High Frequencies Outside the Range of Primary Interest.

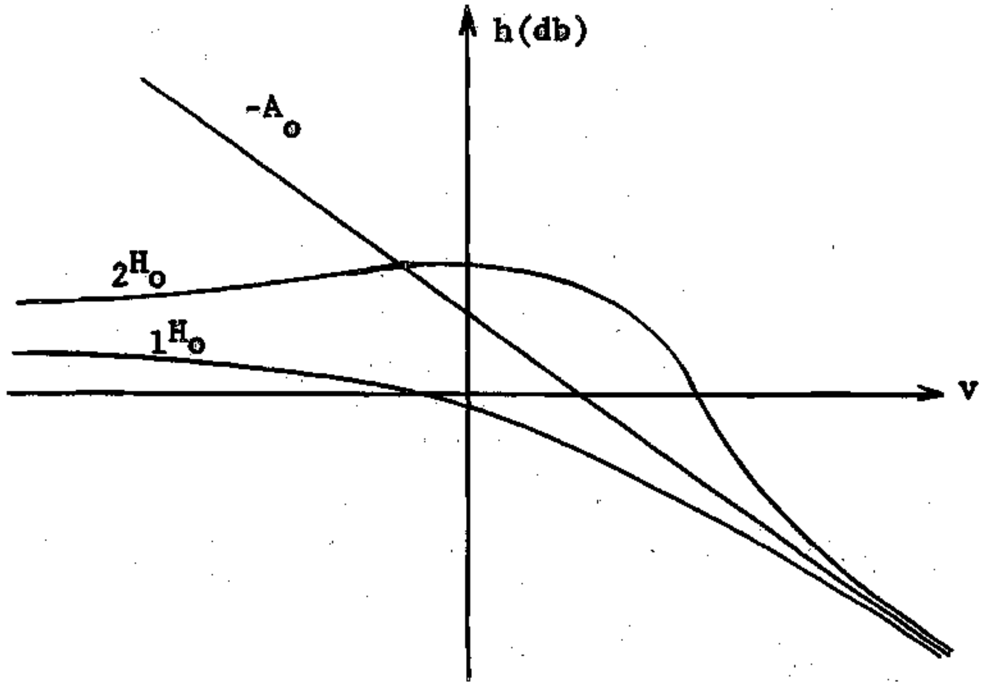

Fig. 24. Intersection of Asymptote with Prescribed Function.

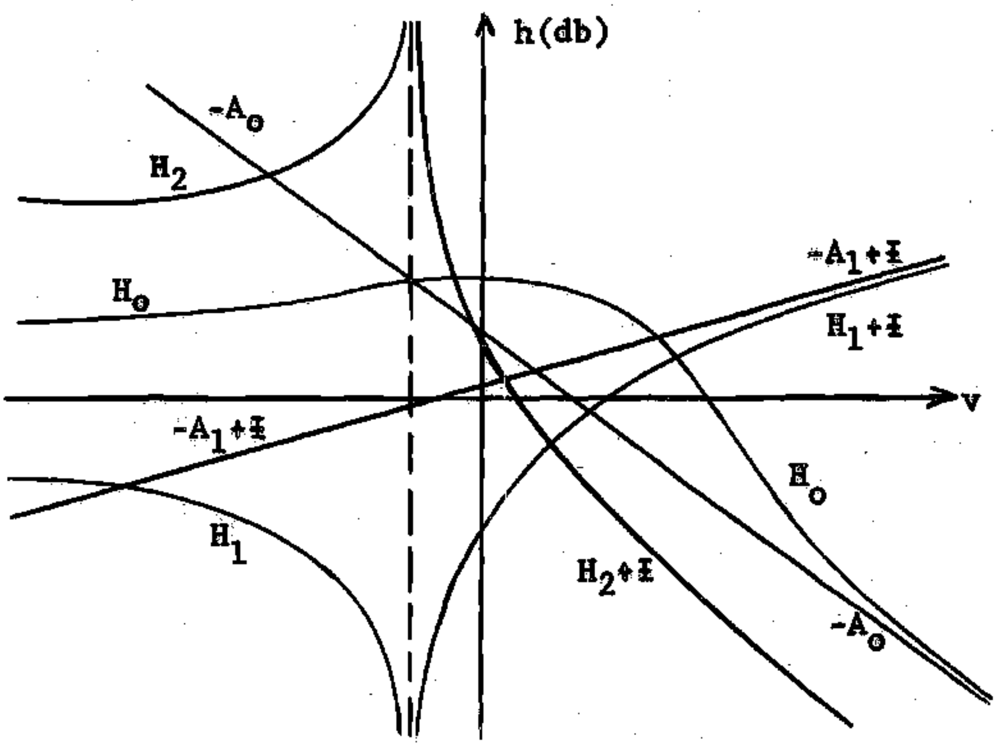

Fig. 25. Typical Curves of Linear Asymptote Method.

the relation $a_k G_k \lesseqgtr 1$, which was used in Chapter III to investigate the convergence of successive approximants, may not apply with the same inequality sign throughout the frequency spectrum. Also the sign of $H_o + A_o$ may change in the range of interest, with a consequent change in the $\pm$ symbol assigned to H_1. Figure 25 portrays a series of typical curves which may occur as a result of an intersection of an asymptote with a prescribed function.

<u>An inherent difficulty of the linear asymptote method</u>.--The major problem presented by this method is that of obtaining an approximant acceptable over the entire frequency range when formally the method deals only with high-frequency asymptotic expressions. For example, suppose that for ω greater than one, G_o actually has the values represented by the convergence limit of

$$G_o = \cfrac{1}{\omega^2 + \cfrac{1}{\omega^2 + \cfrac{}{\omega^2 + \ddots}}}$$

whereas for ω less than one, G_o equals 0.618. The prescribed function and the first six approximants obtained by continued fraction expansion in terms of asymptotes for large ω are sketched in Figure 26. The approximants become very good in the region where H_o is analytically related to its behavior at infinity, but bear no relation to its behavior at low frequencies; and this situation will persist as long as attention

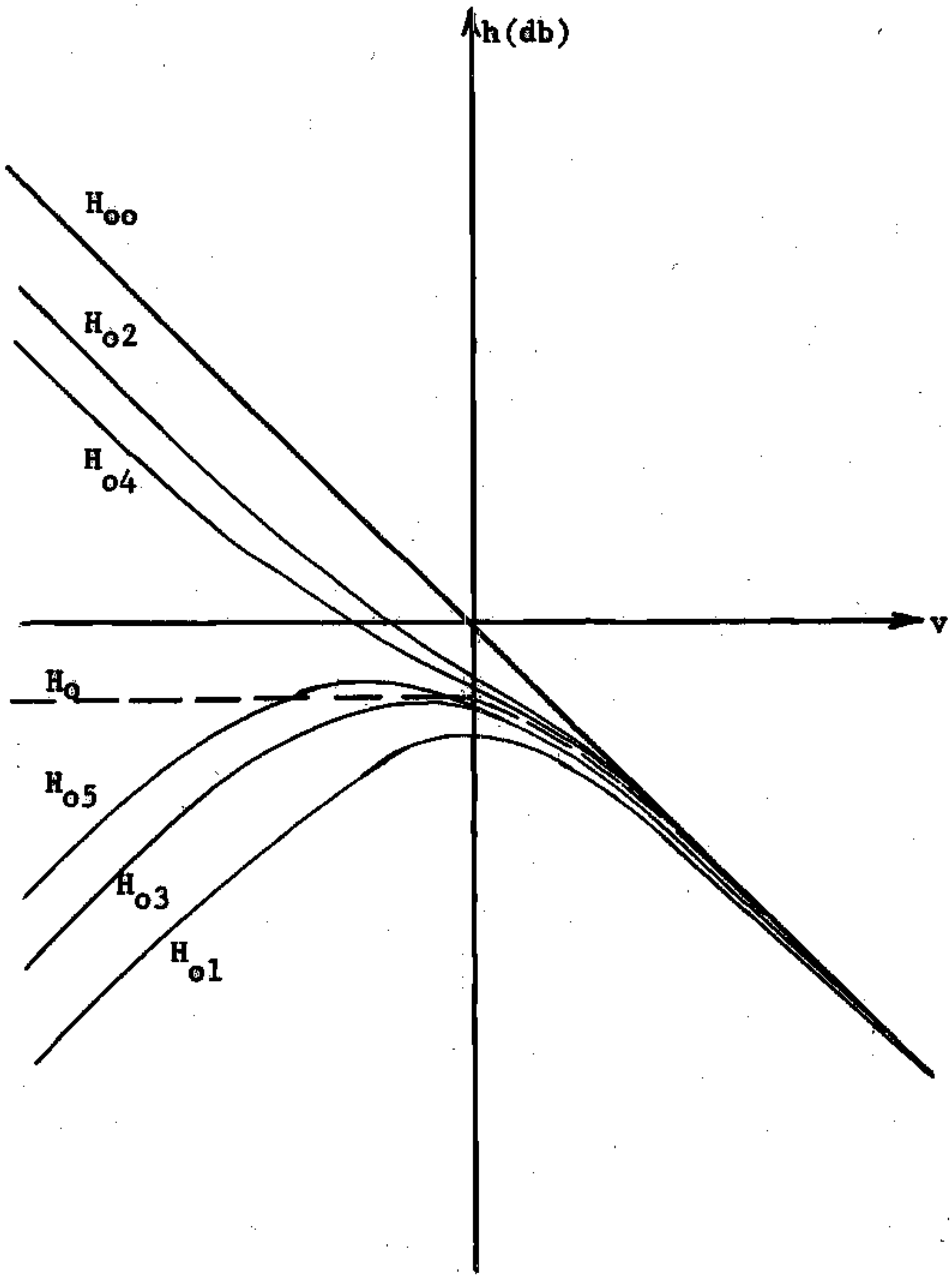

Fig. 26. Approximants for Special Example.

is confined solely to the high-frequency behavior. Since H_o is given graphically or in tabulated form it cannot be supposed that there is any analytic relation between successive portions of the curve.

It may also be noted that in the high-frequency region of Figure 26 successive approximants of higher order are very close to the prescribed curve. In fact, depending upon the prescribed tolerance limits, they may be too close in the sense that they lead to an unnecessarily high order in the resultant final approximant. This suggests the strategy to be developed. It is to start with asymptotic approximations for large ω, but then to shift attention gradually to the left, at the same time adhering to certain conditions which will guarantee that what happens to the right of the current range of interest will not throw the final approximant outside the tolerance bounds.

Effect of precise element location.--First consider the effect of small changes in the location of an asymptotic element. Three possible choices for $-A_o$ are shown in Figure 27, along with the prescribed function and the three resulting remainders, each corresponding to one of the choices for the element. Suppose that H_o has a fourth order zero at infinity. The $-A_o$ curves have the form $K - 40\log \omega$. ${}_1H_1$ is the remainder corresponding to $-{}_1A_o$. Its shape indicates that the full coefficient of ω^4 in the denominator of G_o was not removed, leaving ${}_1G_1$ with a fourth order pole at infinity. The shape of ${}_3H_1$ indicates that the coefficient of ω^4 in ${}_3a_o$ is greater than the coefficient of ω^4 in the denominator of G_o, leaving ${}_3G_1$ with a fourth order pole at infinity

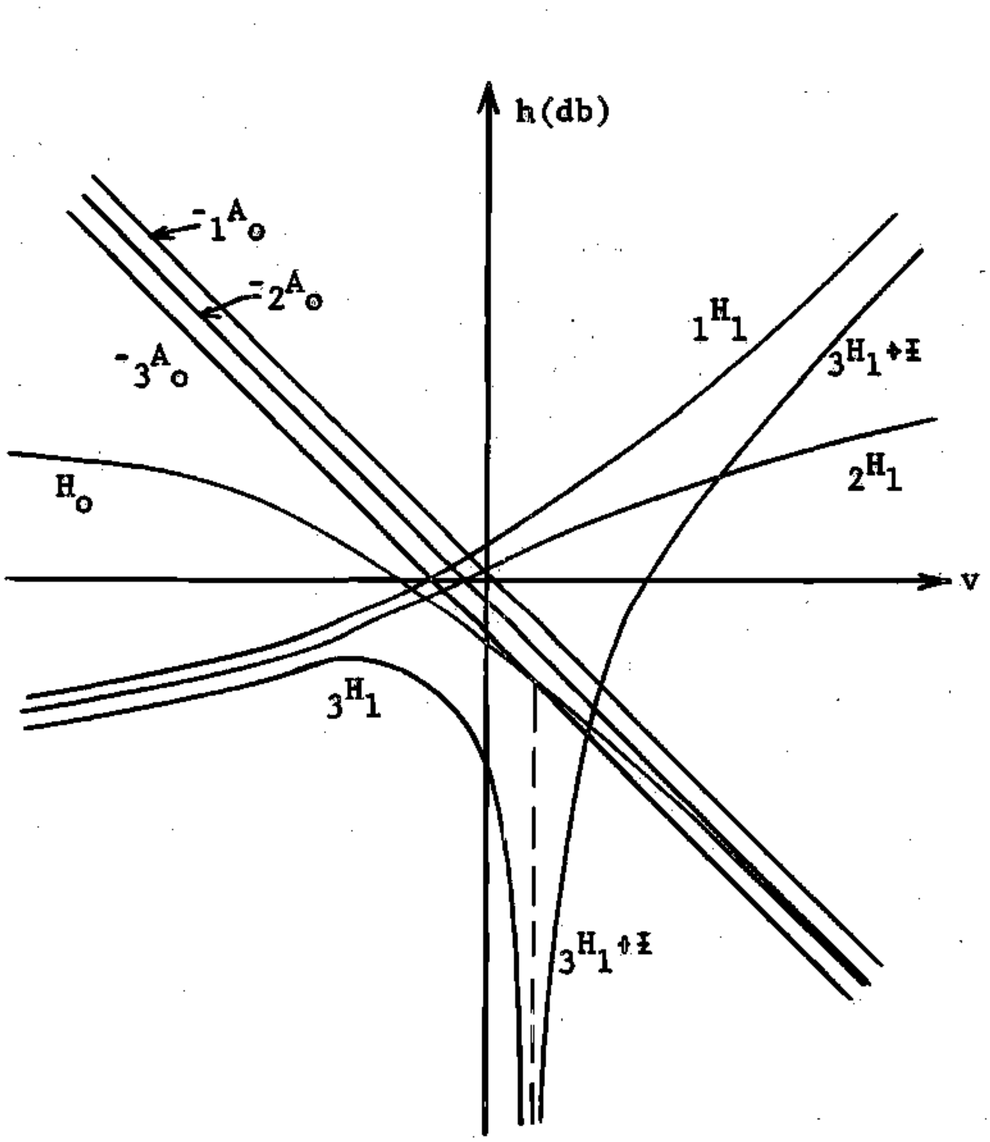

Fig. 27. Effect of Small Changes in Location of Linear Element. (The Subindices Indicate the Particular Remainder H_1 to be Associated with Each Choice for $-A_o$.)

having a negative coefficient. Hence the I symbol is associated with the right hand portion of ${}_3H_1$. The shape of ${}_2H_1$ indicates that exactly the proper coefficient was removed, and that ${}_2H_1$ has a second order pole at infinity remaining. It can be seen that ${}_2H_1$ is the preferable remainder. Otherwise the next step involves removing the residual fourth order coefficient, whether positive or negative, and an unnecessary stage has been included in the procedure.

Remainders with linear asymptotes.--In Figure 27 ${}_2H_1$ is shown having a straight line as its high-frequency asymptote. It is entirely possible that no such H_1 can be found. A remainder that is linear at high frequencies will only occur under special circumstances, as shown below. Equations (43) and (46) give the condition for a linear H_1 as

$$A_o = - H_o + L(H_o + 20p_1v + K_1)$$

in which K_1 may be real or complex. This may be written in an equivalent form,

$$H_o + 20p_1v + K_1 = L(A_o + H_o)$$

For large v, $A_o + H_o$ becomes very small and

$$L(A_o + H_o) = 10 \log(1 - 10^{\frac{1}{10}(A_o + H_o)})$$

$$\approx 10 \log(1 - (1 + \frac{\ln 10}{10}(A_o + H_o) + \ldots))$$

$$\approx I - 10(1 + \log \log e) + 10 \log(A_o + H_o)$$

At the same time H_o approaches $K_o - 20p_ov$, so that

$$K_o + K_1 - 20(p_o-p_1)v \approx I - 10(1 + \log\log e) + 10\log(A_o + H_o)$$

$$C - 2(p_o-p_1)v \approx \log\,(A_o + H_o) \tag{48}$$

$$A_o + H_o \approx \frac{C'}{10^{2(p_o-p_1)v}}$$

Since $A_o + H_o$ becomes very small, p_o must be greater than p_1. C' may be positive or negative.

Equation (48) shows the manner in which $A_o + H_o$ must approach zero as v increases in order for the remainder to have a linear asymptote. If p_1 is not an integer H_1 will become linear, but its slope will not be a multiple of twenty decibels per decade. If $A_o + H_o$ does not approach zero in this exponential manner, H_1 will not approach a linear slope. For example, if $A_o + H_o$ equals $\frac{1}{10v}$ for large v,

$$H_1 = A_o + I + L(-A_o - H_o)$$

$$\approx -K_o + 20p_ov + I - 20 - 10\log\log e - 10\log v$$

which is not linear.

<u>Incorporation of tolerance bounds into the method</u>.--A procedure is needed which will take care of non-linear remainders of the type indicated above, and which will also permit the range in which the successive asymptotic terms $-A_k$ are close to their respective remainders to be extended into

regions of progressively lower frequency. Such a procedure is obtained by considering the tolerance limits prescribed in the problem. Figure 28 depicts a given H_o and corresponding tolerance bounds, along with the asymptote $-A_o$, the remainder H_1, and the tolerance bounds for the remainder. Figures of this type may be simplified if the H_k themselves, and those boundary segments which are not on the same sheet as the succeeding $-A_k$, are omitted. Modifying Figure 28 in this manner leads to Figure 29.

On the latter, two possible choices for $-A_1$ are indicated. Both fall within the tolerance bounds H_1^+ and H_1^- for large ω. In both cases the range in which $-A_1$ falls between the tolerance bounds extends to the left of the range in which $-A_o$ falls between H_o^+ and H_o^-. This means that if the expension is halted after the determination of $-A_1$, H_{o1} falls within prescribed tolerances over a greater range than H_{oo}. As to differences between the two choices, $-{}_1A_1$ falls between H_1^+ and H_1^- over the greater range, while $-{}_2A_1$ is closer to the desirable band at very low frequencies. Before making the selection, the advisability of inverting H_1 prior to the selection must be considered.

Effect of inverting remainder before expanding.--The effect of inverting a remainder before continuing the procedure is best illustrated in a simple example in the algebraic domain. Let

$$G_o \approx \frac{1}{\omega^4 + \omega^2 + 1}$$

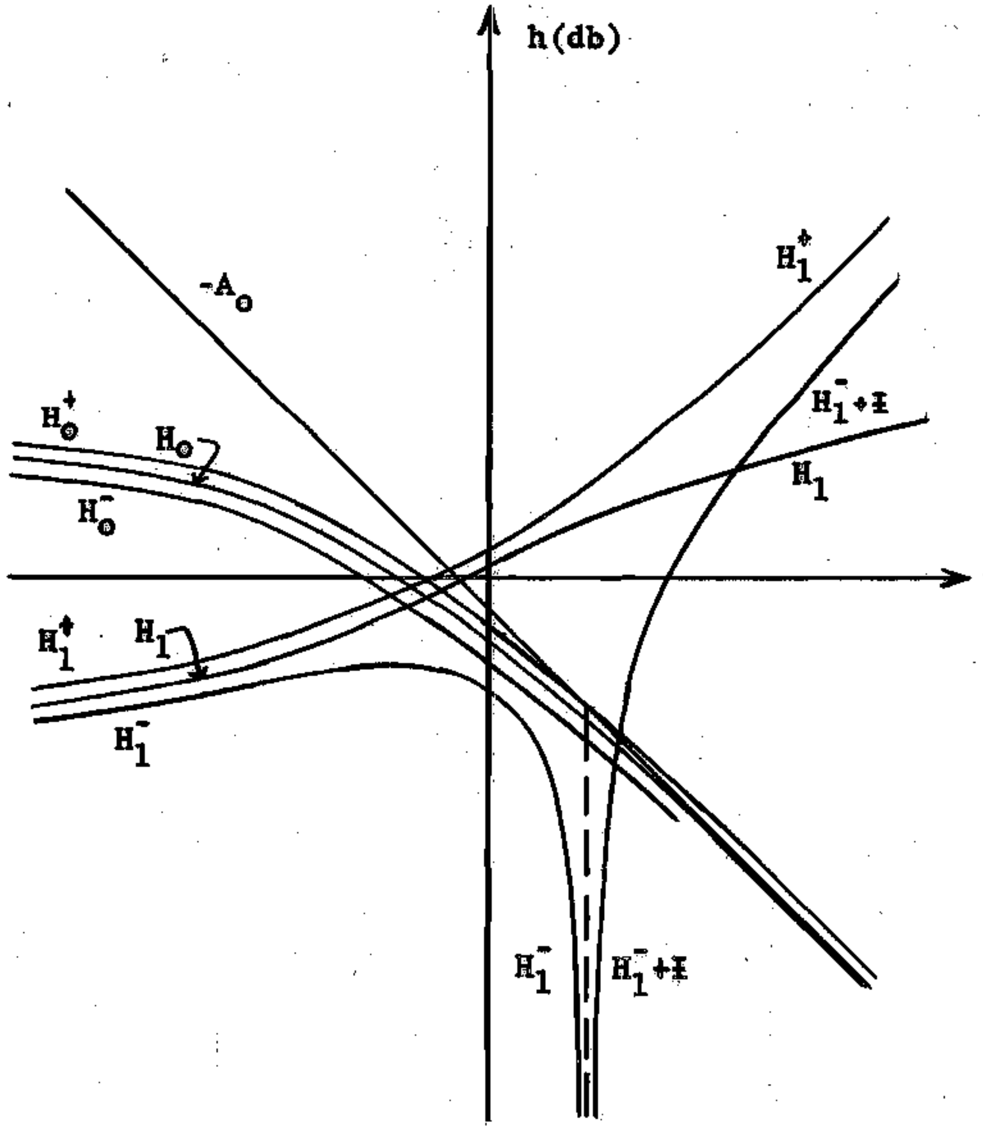

Fig. 28. Tolerance Bounds in the Linear Asymptote Method.

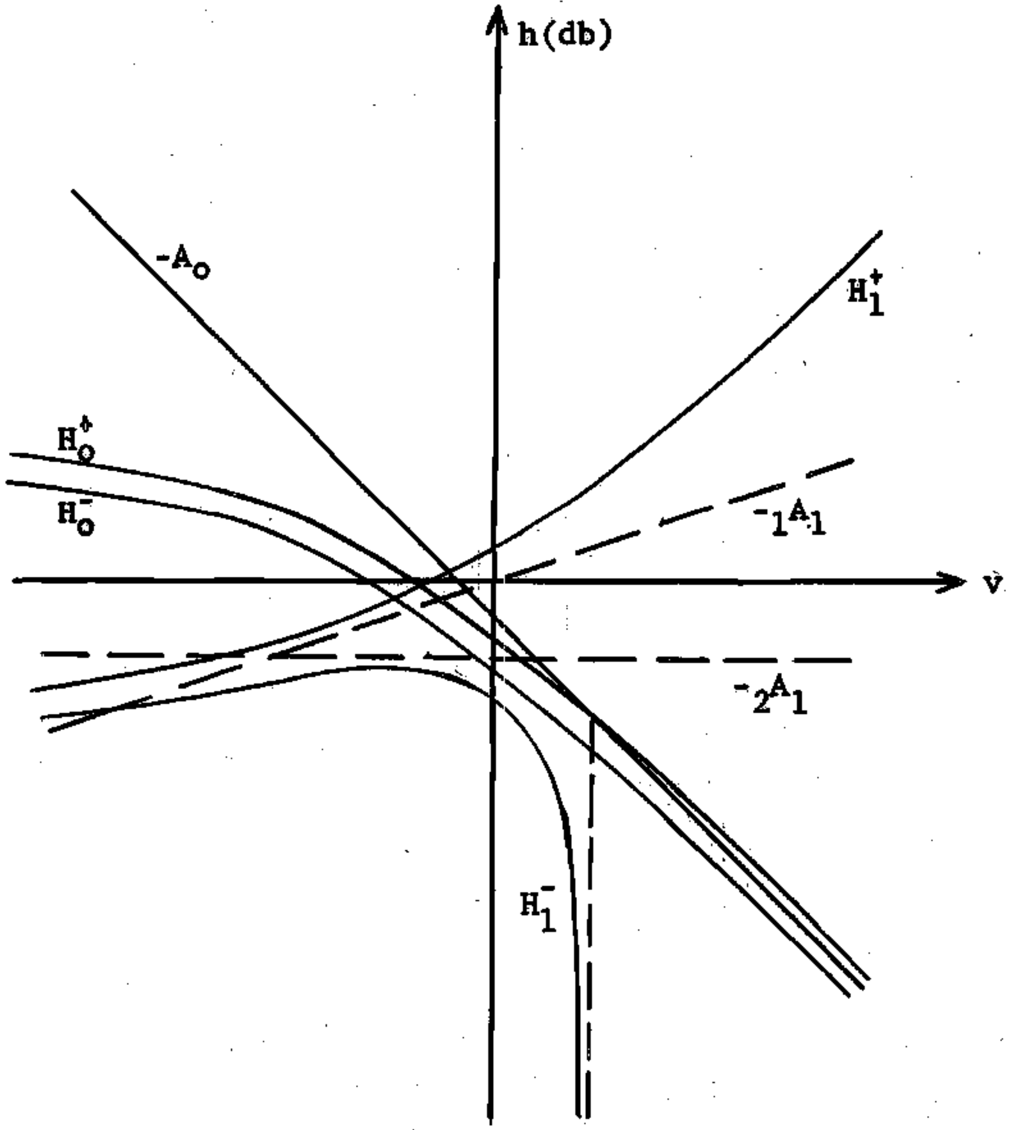

Fig. 29. Essential Curves for Linear Asymptote Method. (A Segment of the H_1^- Bound has been Replaced by $-\infty$. Two Possible Choices for the Next Element are $-_1A_1$ and $-_2A_1$.)

From equation (4) G_1 is determined to be

$$G_1 \approx \omega^4 + \omega^2 + 1 - \omega^4 = \omega^2 + 1$$

If G_1 is not inverted, and the normal procedure is used to find G_2,

$$G_2 = \frac{1}{G_1} - a_1 \approx \frac{1}{\omega^2 + 1} - \frac{1}{\omega^2} = \frac{-1}{\omega^4 + \omega^2}$$

Continuing in a similar manner,

$$G_3 \approx - \frac{\omega^4 + \omega^2}{1} - - \omega^4 = - \omega^2$$

$$G_4 \approx - \frac{1}{\omega^2} - \left(- \frac{1}{\omega^2}\right) = 0$$

Recombing the elements to form G_{o3};

$$G_{o3} = \cfrac{1}{\omega^4 + \cfrac{1}{\cfrac{1}{\omega^2} + \cfrac{1}{-\omega^4 + \cfrac{1}{-\cfrac{1}{\omega^2}}}}} = \frac{1}{\omega^4 + \omega^2 + 1}$$

If, on the other hand, G_1 is inverted before G_2 is determined, the calculations are:

$$G_2 = \frac{1}{\frac{1}{G_1}} - a_1 = G_1 - a_1 \approx \omega^2 + 1 - \omega^2 = 1$$

$$G_3 \approx \frac{1}{1} - 1 = 0$$

$$G_{o2} = \frac{1}{\omega^4 + \omega^2 + \frac{1}{1}} = \frac{1}{\omega^4 + \omega^2 + 1}$$

The principal result is that in the second case the final approximant was reached in one less stage, thereby simplifying the calculations.

It is not always true that inversion reduces the number of stages. For example, in the case of

$$G_o \approx \frac{\omega^2 + 1}{\omega^6 + \omega^4 + \omega^2}$$

the normal procedure achieves the desired final approximant in fewer stages than the inverting procedure does.

Inversion is accomplished in the logarithmic domain simply by replacing H_k with $-H_k$ before selecting $-A_k$. Its effect on the expansion in the algebraic domain is to replace

$\frac{1}{a_k + G_{k+1}}$ with $a_k + G_{k+1}$.

Possibility of obscuring an important variation in the prescribed function.-- A second important consideration in deciding whether or not to invert a remainder before expanding it in continued fraction form is illustrated in the following example. Let

$$G_o \approx \frac{\omega^2}{\omega^6 + \omega^4 + \omega^2 + .01} = \frac{1}{\omega^4 + \omega^2 + 1 + \frac{.01}{\omega^2}}$$

As in the preceding section, a_o equals ω^4, and

$$G_1 \approx \omega^2 + 1 + \frac{.01}{\omega^2}$$

Inverting and expanding,

$$G_2 \approx 1 + \frac{.01}{\omega^2}$$

From this point, G_2 may be expanded in the normal manner, or it may be inverted again and expanded; in both cases the desired final approximant will be achieved, although in the second instance it will be reached in fewer stages.

Suppose, however, that G_1 were expanded without inversion.

$$G_2 \approx \frac{1}{\omega^2 + 1 + \frac{.01}{\omega^2}} - \frac{1}{\omega^2} = \frac{-1 - \frac{.01}{\omega^2}}{\omega^4 + \omega^2 + .01}$$

$$\approx -\frac{1}{\omega^4 + \omega^2}\left(\frac{\omega^4 + 1.01\,\omega^2 + .01}{\omega^4 + \omega^2 + .01}\right)$$

The factor in parentheses is approximately equal to unity at all frequencies, and is never greater than $1.0083\dot{3}$. In terms of decibels the maximum deviation of G_2 from its value with the parenthetical factor omitted is 0.036 db. It is easy to see that a factor of this type could readily be lost from view in the logarithmic calculations. If in this case the factor is lost, the expansion is the same as in the preceding section, and leads to

$$G_{o3} = \frac{1}{\omega^4 + \omega^2 + 1}$$

instead of the desired approximant; and a significant variation at low frequencies has been lost.

Criterion for decision on inversion.--A policy for deciding whether or not to invert a given H_k is obtained by considering the desirable properties of the remainder H_{k+1}. The remainder should not obscure significant variations in the prescribed function, and it should lead to an expansion of fewer stages. Table 1 is a very useful aid in making a determination on inversion without going into detailed calculations. It is based on equation (12), and provides the means for making a rough sketch of H_{k+1} for both options, normal and inverted.

Suppose that some H_{on} is an acceptable approximant. H_n must then approximate a straight line, so that $-A_n$, the logarithmic expression of the last element, can fall close to H_n throughout the frequency spectrum. This suggests that it is desirable for successive H_k to have progressively smaller differences between their high-frequency slopes and their low-frequency slopes. To be more specific, let s_k^o and s_k^i be the slopes of H_k at zero and infinity respectively. Table 1 shows that as $-A_k$ approaches H_k at infinity, H_{k+1} is less than either $-H_k$ or $A_k + \mathrm{I}$ by increasingly greater amounts. s_{k+1}^i is therefore less than $-s_k^i$; let s_{k+1}^i equal $-s_k^i - N$. At the low frequency end the behavior of H_{k+1} depends on whether $\mathrm{Re}(-A_k)$ is greater or less than $\mathrm{Re}(H_k)$. If s_k^i is greater than s_k^o, $-A_k$, which has, of course, the constant slope s_k^i, falls far below H_k (or $H_k + \mathrm{I}$) at low frequencies, and H_{k+1}, which then approximately equals $A_k + \mathrm{I}$, has the slope, s_{k+1}^o equals $-s_k^i$. The difference between s_{k+1}^o and s_{k+1}^i is N. If on the other hand s_k^i is less than s_k^o, $-A_k$ falls above H_k (or $H_k + \mathrm{I}$), and s_{k+1}^o equals $-s_k^o$. The difference between s_{k+1}^o and s_{k+1}^i is then equal to $N - s_k^o + s_k^i$, which is less than N, and usually smaller in magnitude.

Table 1. Approximate Dependence of Remainder on Location of Preceding Element.

a. $-A_k$ and H_k on the same sheet.

Location of Element	Location of Remainder*
$-A_k \ll H_k$	$H_{k+1} \approx A_k + \text{Ɨ}$
$-A_k < H_k$	$H_{k+1} < A_k + \text{Ɨ}$
$-A_k = H_k$	$H_{k+1} = -\infty$
$-A_k > H_k$	$H_{k+1} < -H_k$
$-A_k \gg H_k$	$H_{k+1} \approx -H_k$

b. $-A_k$ and H_k on opposite sheets.

Location of Element	Location of Remainder*
$-A_k \ll H_k + \text{Ɨ}$	$H_{k+1} \approx A_k + \text{Ɨ}$
$-A_k < H_k + \text{Ɨ}$	$H_{k+1} > A_k + \text{Ɨ}$
	$< A_k + \text{Ɨ} + 3.01$
$-A_k = H_k + \text{Ɨ}$	$H_{k+1} = A_k + \text{Ɨ} + 3.01$
	$= -H_k + 3.01$
$-A_k > H_k + \text{Ɨ}$	$H_{k+1} < -H_k + 3.01$
	$> -H_k$
$-A_k \gg H_k + \text{Ɨ}$	$H_{k+1} \approx -H_k$

*Note: H_{k+1} is on the same sheet as the quantity with which it is compared.

The foregoing analysis suggests that the following procedure be adopted:

If $-A_k$ falls above $H_k(+I)$ at low frequencies, expand in the normal manner. If $-A_k$ falls below $H_k(+I)$ at low frequencies, invert H_k and then expand.

Fortunately, this rule is also good from the viewpoint of conserving significant variations. From Table 1, if $-A_k$ falls far above $H_k(+I)$, H_{k+1} is approximately equal to $-H_k$ and thus preserves variations in H_k in the region where the inequality holds. On the other hand, if $-A_k$ is much less than $H_k(+I)$, H_{k+1} is approximately equal to $A_k + I$, which is linear; hence significant variations from H_k may be lost from view in ranges where this inequality holds. The example of the preceding section falls in this latter category at low frequencies.

<u>Application of inversion criterion and selection of element</u>.--Return now to a consideration of the problem illustrated in Figure 29. Here two choices for $-A_1$ are available: $-{}_1A_1$ has a slope of 20 decibels per decade and $-{}_2A_1$ has zero slope. In addition a decision is to be made whether or not to invert H_1 before proceeding. Consider first the case of $-{}_2A_1$. Whether H_1 is inverted or not, the asymptotic behavior of H_2^+ and H_2^- for small ω will have zero slope. If $-A_2$, the next succeeding element, were to be the last element in the expansion, it would also have a zero slope. But if such a $-A_2$ were possible it would also be possible to find a value for $-{}_2A_1$ that would fit between the tolerance bounds for all frequencies, and the figure indicates that such a choice does not exist. Thus if $-{}_2A_1$, inverted or not, is chosen, there must

be at least two additional stages remaining before the expansion procedure is completed.

Suppose that $-{}_1A_1$ is chosen without inverting. For low frequencies $-{}_1A_1$ is very much less than H_1^+ and H_1^-. Table 1 indicates that in this range H_2^+ and H_2^- will be approximately equal to ${}_1A_1 + I$. Not only do low-frequency variations of $H_1^{\pm}$ disappear in $H_2^{\pm}$, but the band between H_2^+ and H_2^- practically vanishes at low frequencies, so that selection of $-A_2$ is very restricted. Also $-A_2$ cannot be the last element because it and all preceding elements have non-zero slopes; since H_0 has zero slope for small ω, no expansion solution can be found without at least one element with zero slope, that is, without at least one constant in the continued fraction expansion.

The remaining option is to invert H_1 and test $-{}_1A_1$. The pertinent curves appear in Figure 30. The curves H_2^+ and H_2^-, which may be sketched with the aid of Table 1, indicate that there is a chance of finding a $-A_2$ which will fit between the prescribed limits for all frequencies. Only an actual calculation with numerical values will reveal whether or not such an element is indeed possible. Some adjustment of $-{}_1A_1$ may be needed, particularly to keep it far enough from H_1^- so that the tendency of H_2^+ to dip down in the middle of the frequency range will be limited.

A simplification in the calculation of tolerance bounds.--The preceding figures indicate that where tolerance bounds are incorporated in the linear asymptote method, they frequently change from one sheet to another and present a confusing aspect, especially when upper and lower bounds

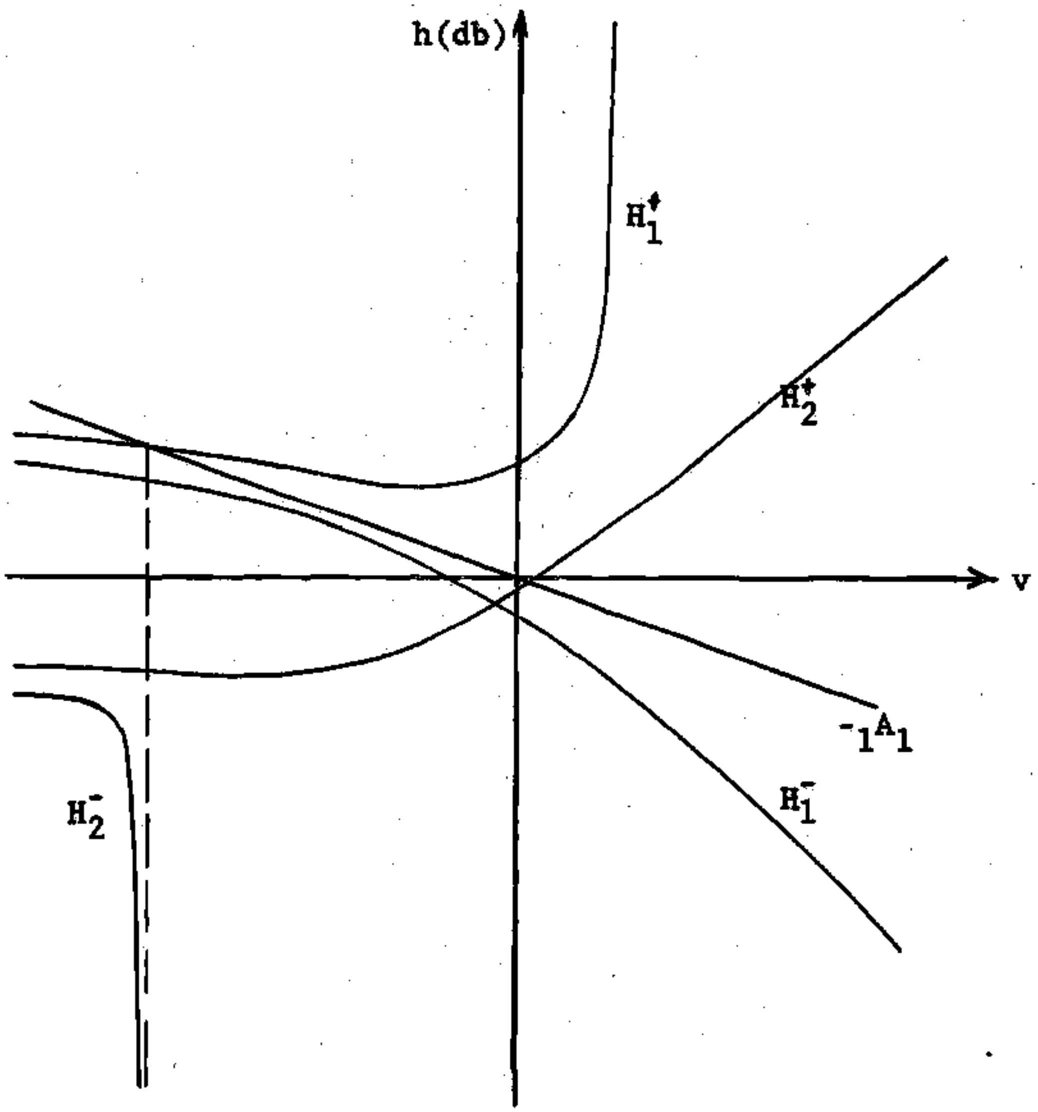

Fig. 30. Inverted Tolerance Bounds, Linear Element, and Remainder Bounds, Representing Optimum Choice of Inversion Option and Element.

are on opposite sheets. A simplification as well as a saving of computational labor would result if certain segments of the bounds could be disregarded with the assurance that the final result will still satisfy the prescribed tolerances. In Chapter IV such omissions were permissible because of the requirement that $-A_k$ be greater or less than H_k^+ or H_k^- in a systematic manner. Now that $-A_k$ are limited to linear functions and may therefore intercept both bounds, it can no longer be assumed that these omissions will not introduce errors.

Figure 31 shows the effects on tolerance bounds of a particular choice of $1/a_o$ at three different frequencies. In all three instances the band in which it is desirable for $1/a_1$ to fall is shown shaded. Now suppose that because of the shapes of G_1^+ and G_1^- it is desirable to invert G_1 before proceeding. Letting

$$g_1' = \frac{1}{g_1}, \quad G_1^{+'} = \frac{1}{G_1^-} \quad , \text{ and } G_1^{-'} = \frac{1}{G_1^+} \quad ,$$

and again shading the segments of g_1' most desirable for the location of $1/a_1$, the situation at the three frequencies in question appears as shown in Figure 32. Note first that the desirable segment for the location of $1/a_1$ in the case of ω_3 now extends along g_1' out to plus and minus infinity instead of through zero. A comparison of the location of the tolerance limits at the three frequencies shows that the half-axis on which g_1' is negative is the region of interest for the location of $1/a_1$. This corresponds to the logarithmic sheet on which H_1 carries the $\mp$ symbol. Suppose further that in order to simplify the calculations

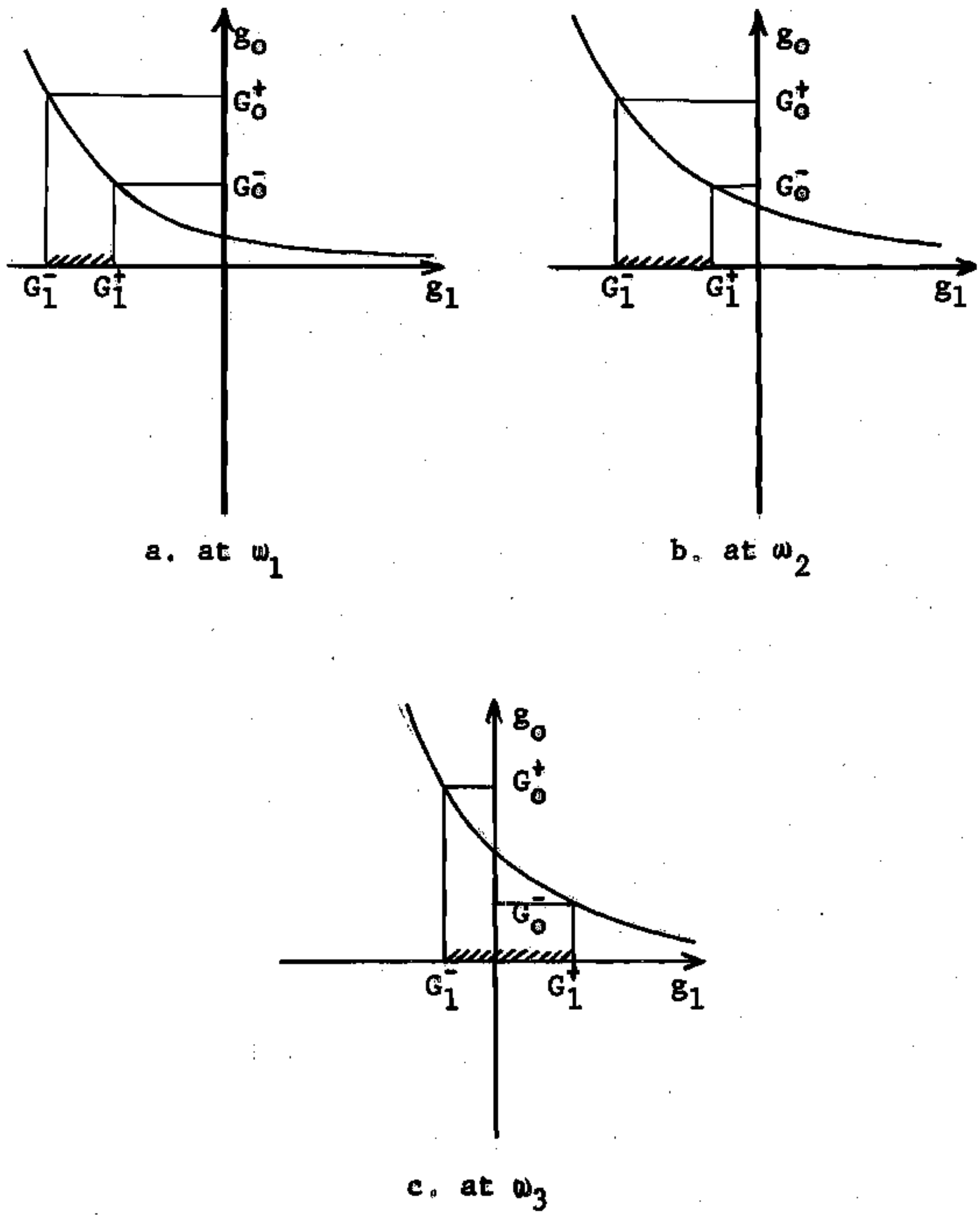

Fig. 31. Location of Tolerance Bounds at Three Frequencies.

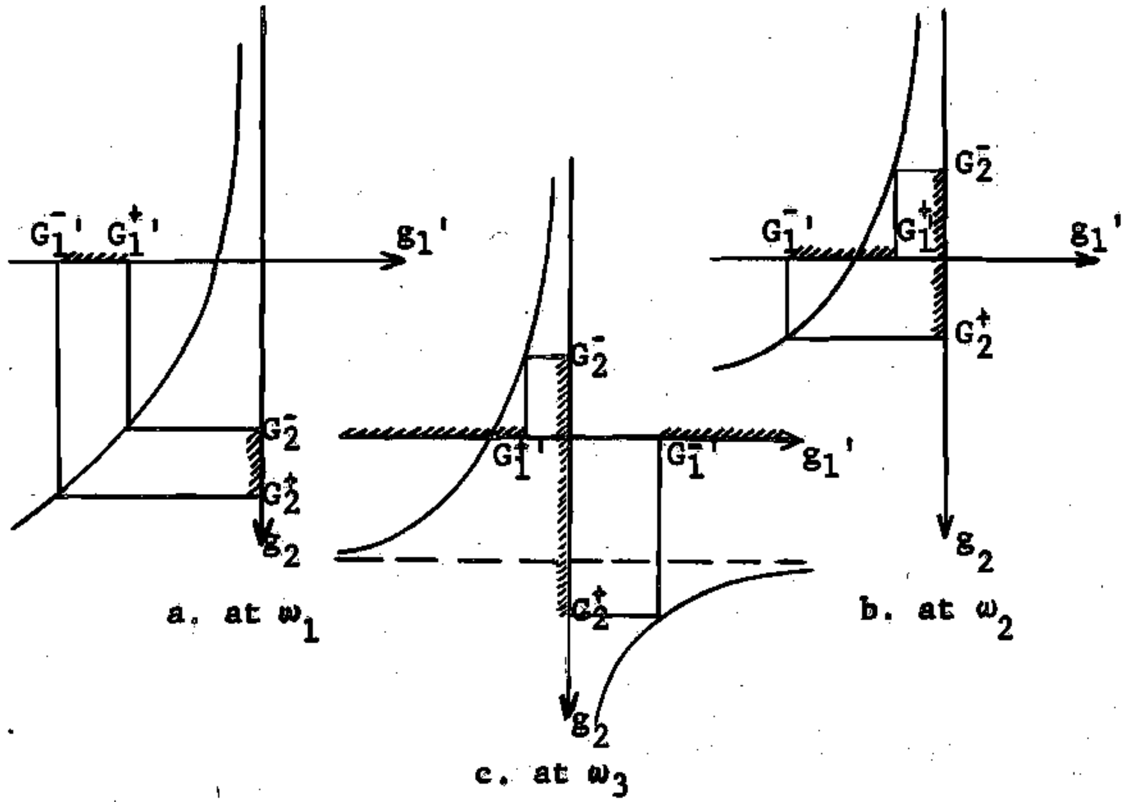

Fig. 32. Inversion of g_1 and Effect of Element Choice on Bounds for g_2.

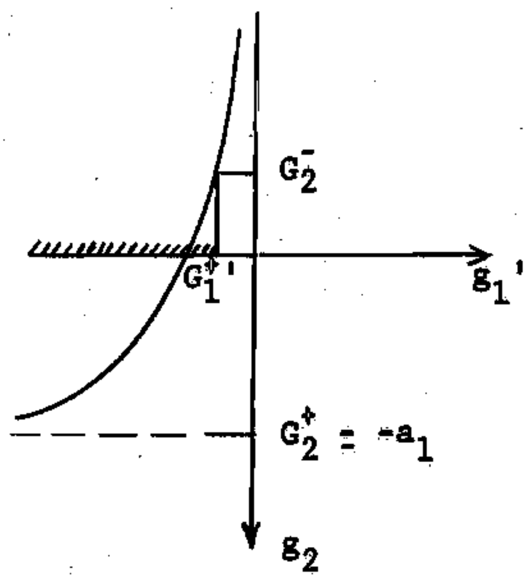

Fig. 33. Modified G_2^{+} Obtained by Assuming $G_1^{-}{}'$ at Infinity.

$G_1^{-'}$ was dropped from further consideration in the region of ω_3 as being on the opposite half-axis from that on which it was planned to locate $1/a_1$ (this follows the practice of Chapter IV). Now observe the tolerance bounds on g_2. Evidently the half-axis of interest is the positive one. In the case of ω_3, G_2^+ sets an upper limit of $1/a_2$. If, however, $G_1^{-'}$ had been dropped from further consideration, as suggested above, the location of G_2^+ would not be known, and $1/a_2$ might be chosen to fall outside the desirable band although assumed to be inside it.

It would appear at first glance, then, that it is necessary to carry the full calculations for both tolerance bounds throughout the entire problem. A compromise solution, however, that appears to be satisfactory in practical cases, and which reduces the number of calculations, is to assume $G_1^{-'}$, in such situations as that of Figure 32, to be at infinity in the region of ω_3. The effect on g_2 is shown in Figure 33. The desirable region for $1/a_2$ has been reduced a little, but in practice this does not prove to be unduly restrictive. The calculation of G_2^+ is simple; in the logarithmic domain it is

$$H_2^+ = \bar{I} + A_1 + L(-\infty) = \bar{I} + A_1, \text{ a straight line.}$$

The sequence, Figures 34 and 35, illustrate the effect on logarithmic curves of adopting the above convention. Figure 35 shows that the H_2^+ bound based on the assumed H_1^- does not restrict the choice of $-A_2$.

<u>Summary</u>.--The procedure for expansion in terms of linear high-frequency asymptotes is summarized below. The procedure is applied to the prescribed

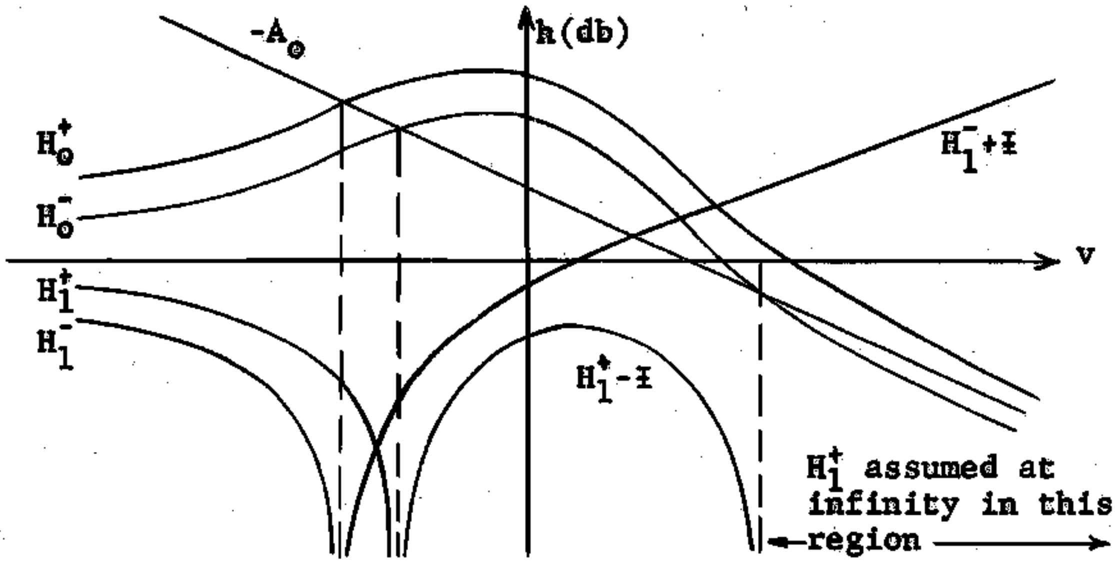

Fig. 34. Convention for Location of H_1^+.

(In order to place $-A_1 + I$ above $H_1 + I$ at low frequencies, H_1 is inverted as shown in Figure 35. This inverts the bounds and exchanges their superscripts.)

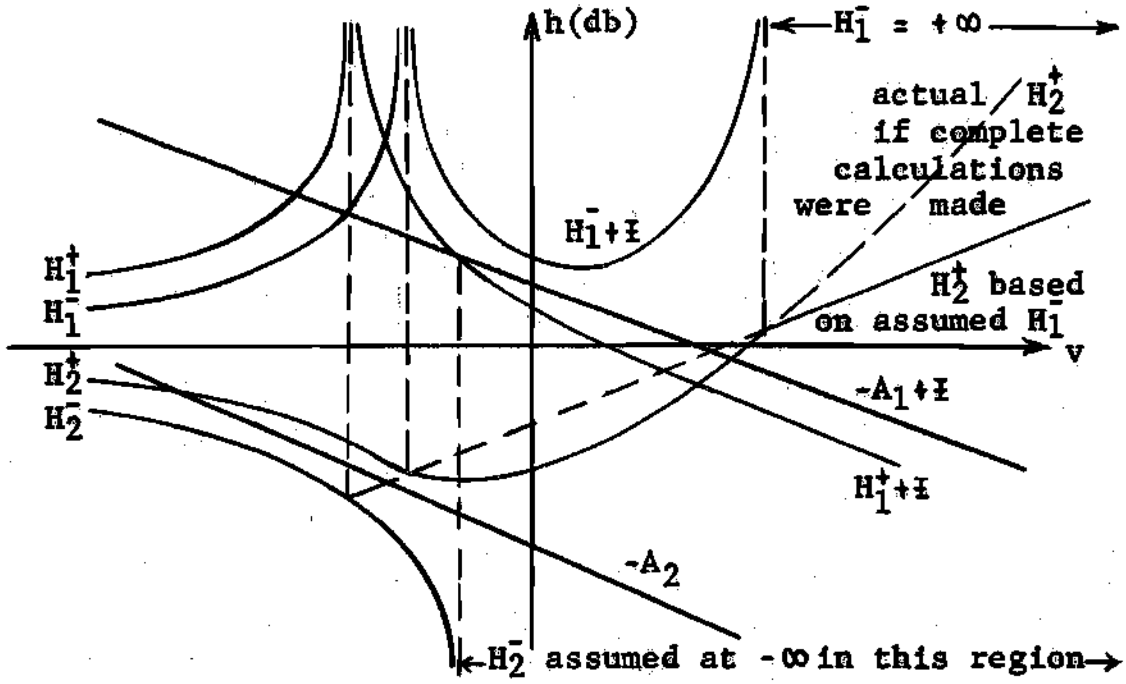

Fig. 35. Effect of Convention on Tolerance Bound in Succeeding Stage.

function and to each successive remainder in the same manner.

(1) Let H_k and its tolerance bounds, H_k^+ and H_k^-, be given. A linear function, $-A_k$, is selected that approaches H_k asymptotically at high frequencies, and falls between the tolerance bounds at high frequencies. $-A_k$ must have a slope equal to an integral multiple (positive, negative, or zero) of twenty decibels per decade. If H_k is complex at high frequencies, so is $-A_k$.

There are two ways to handle the tolerance bounds. One is to carry only the bounds forward from stage to stage and ignore the actual prescribed function. The other is to carry the prescribed function and a few key points on the tolerance bounds forward from stage to stage. In the latter method the prescribed function or any corresponding remainder may be altered slightly, within the tolerance limits, to produce desirable behavior in the succeeding remainder. The latter method may often require fewer computations than the former, but it tends to produce unnecessarily close approximation at high frequencies.

(2) A decision is made on whether or not to invert H_k before calculating H_{k+1}. The decision is based on placing $-A_k$ above H_k or H_{k+1} in regions where the separation between them is great, and on minimizing the difference between the high-frequency and low-frequency slopes of H_{k+1}. Table 1 is an aid in making this determination.

(3) A fine adjustment in the location of $-A_k$ may be made to provide the longest range of frequencies in which a linear $-A_{k+1}$ could fall between H_{k+1}^+ and H_{k+1}^-, or to produce the longest range of linear

behavior in H_{k+1}. Equations (43) or (46), pertinent to the production of linear remainders, may be useful in accomplishing the latter objective.

(4) H^+_{k+1} and H^-_{k+1} are calculated point by point. Equation (12), in either form as convenient, and the tables of L(u) in Appendix C, are used, keeping in mind that H^+_{k+1} devolves from H^-_k, and H^-_{k+1} from H^+_k. Alternatively H_{k+1} only is calculated, together with a few critical points on the tolerance bounds.

(5) Where a tolerance bound of H_{k+1} goes off the logarithmic sheet on which $-A_{k+1}$ will be located, in a region where $-A_{k+1}$ will fall between the bounds, the bound may be replaced by $\pm\infty$ as appropriate. This convention is designed merely to simplify calculations.

(6) H_{k+1} or its tolerance bounds having been established, the procedure is repeated commencing as in paragraph (1).

(7) If $-A_n$ falls within the tolerance bounds for H_n throughout the entire frequency spectrum the expansion procedure is halted. G_{on} is the required approximant, and is calculated as follows;

$$G_{on} = \cfrac{1}{10^{\frac{A_o}{10}} + \cfrac{1}{10^{\frac{A_1}{10}} + \cdots + 10^{\frac{A_{k-1}}{10}} + \cfrac{1}{10^{\frac{A_k}{10}} + \cfrac{1}{\ddots + 10^{\frac{A_n}{10}}}}}} \tag{49}$$

except in cases where inversions were made. Suppose that H_k had been inverted in the expansion process. The formula for G_{on} would then be

$$G_{on} = \cfrac{1}{10^{\frac{A_o}{10}} + \cfrac{1}{10^{\frac{A_1}{10}} + \cdots + 10^{\frac{A_{k-1}}{10}} + 10^{\frac{A_k}{10}} + \cfrac{1}{\ddots + 10^{\frac{A_n}{10}}}}} \tag{50}$$

Example 4, Appendix A, is a numerical illustration of the method of approximation with linear elements which are the high-frequency asymptotes of corresponding remainders.

CHAPTER VI

APPROXIMATION WITH OTHER ASYMPTOTIC FUNCTIONS

In the preceding chapter succeeding elements in the continued fraction expansion of $G_o(\omega^2)$ were chosen so that, within prescribed tolerances, their reciprocals were the asymptotes (in the logarithmic domain) of the successive remainders at high frequencies. The elements were accordingly restricted to the form $b_k\omega^{2p_k}$, which gave them the convenient linear form on logarithmic scales. Neither the restriction of the form nor the requirement that the asymptotic behavior occur at infinity are necessary. The class from which the elements are selected could be extended to include those tabulated in Appendix B, or any more general class of rational functions of the variable ω^2. The elements may also be chosen to exhibit their asymptotic behavior at zero frequency, at some intermediate frequency, at different frequencies in successive stages, or at more than one frequency in each stage. In this chapter several alternative asymptotic methods differing from that propounded in Chapter V are considered.

Linear elements asymptotic at zero frequency.--In the method of continued fraction expansion with linear elements asymptotic at infinity, two different approaches were advanced. In one $-A_k$ was chosen to approach H_k in such a manner that H_{k+1} was essentially linear over a greater range than H_k. In the other attention was centered on the bounds H_k^+ and H_k^-, rather than on the function H_k. The former approach tends to produce a final approximant that is much closer to H_o at high frequencies than at low ones. The same tendency is present in the latter approach, but is much less marked.

This suggests that if primary interest is centered on the behavior of the prescribed function at other than high frequencies, it might be well to develop the continued fraction expansion in terms of elements asymptotic at or near the frequency of greatest interest. In particular if this frequency is zero, $-A_k$ may be chosen as the low frequency linear asymptotes of corresponding remainders. Instead of starting at the right and working the region of satisfactory approximation toward the left, we may start at the left and work to the right. In all other respects the method is identical with that of Chapter V.

<u>Linear elements asymptotic at an intermediate frequency</u>.--In the case of expansion in terms of functions asymptotic at some intermediate frequency the situation is more complicated. First of all, since G_o is finite and non-zero at all intermediate frequencies, the first element removed is a constant. The logarithmic curves appear as in Figure 36, where ω_m is the intermediate frequency of interest. The figure shows that H_1 changes from the complex sheet to the real sheet at v equals log ω_m, going through minus infinity. This is a result of the fact that G_1 has a zero at ω_m and changes sign at that frequency. Evidently no linear functions can be found to approach H_1 asymptotically at log ω_m. Functions of the form

$$K(+ I) \pm 10p_k \log \left(\frac{\omega^2}{\omega_m^2} - 1\right)$$

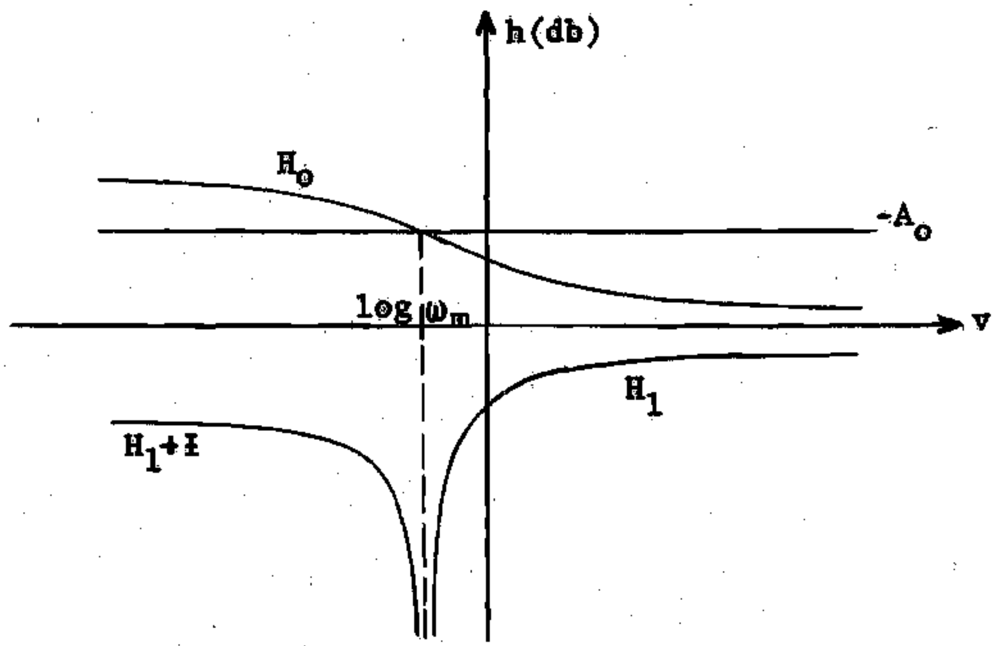

Fig. 36. Singularity Produced by Removal of First Element with Curves Plotted to Normal Frequency Scale.

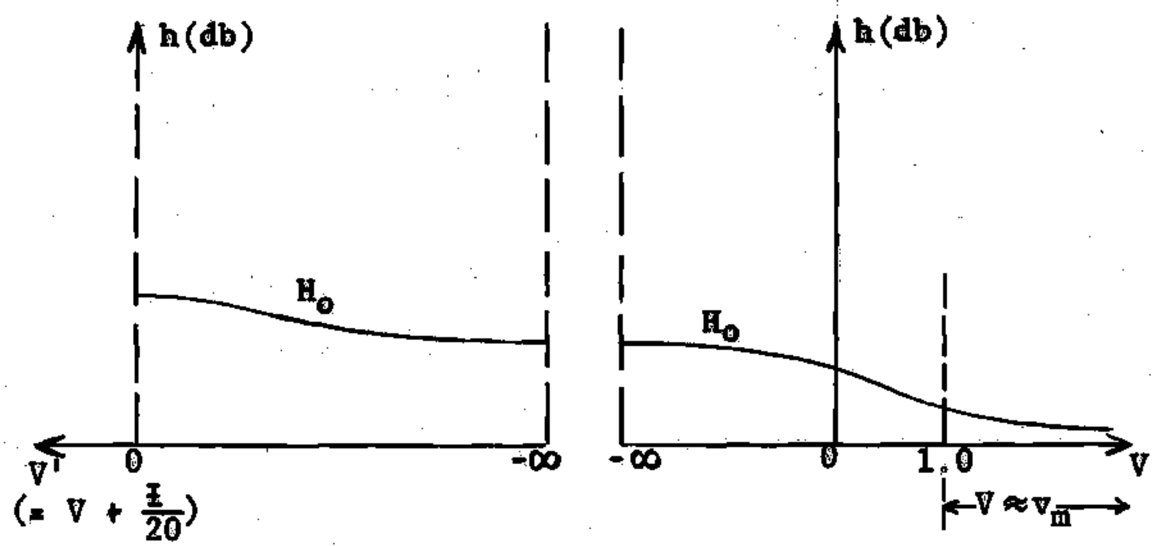

Fig. 37. Replot of H_o with V as Abscissas.

are needed; one such function would approach H_1 in the desired manner at log ω_m. A difficulty is posed by the fact that tables of such factors would have to be carefully computed for very small increments of ω in the vicinity of ω_m so that the shape of the selected function would approximate the curved shape of H_1 in an asymptotic manner as ω approached ω_m; and even with such tables it is inherently harder to match two curves together than to match a straight line with an asymptotically linear function.

An alternative procedure is to replot H_o at the outset to the new frequency scale,

$$V = \frac{\log \lambda^2}{2} = \frac{\log \left(\frac{\omega^2}{\omega_m^2} - 1\right)}{2} \tag{51}$$

Letting $v_m = \log \frac{\omega}{\omega_m}$,

$$V = \frac{I + L(20v_m)}{20} = v_m + \frac{L(-20v_m)}{20} \tag{52}$$

From equation (51), when ω is greater than ω_m, v_m is positive, λ^2 is positive, and V is real. When ω is less than ω_m, however, v_m is negative, λ^2 is negative, and V is complex. Equation (52) shows that when V is complex, its imaginary part is $I/20$. As ω^2 varies from zero to infinity, λ^2 varies from -1 to infinity, and V varies from $0 + I$ to $-\infty + I$ and from $-\infty$ to $+\infty$. Thus V may have any real value, but only complex values with negative real parts. This means that the replot of H_o will appear on both sheets of V as indicated in Figure 37.

Let $V \pm \frac{I}{20} = V'$ for simplification in labelling axes; thus V' is the real part of V when the latter is complex. The replotting of H_o is simplified by the fact that for large positive v_m, V is approximately equal to v_m, and the plot of H_o is not altered in this region. For example, when v_m equals 1.0, V equals 0.9978; if the shape of H_o at this point indicates that a lateral shift of the curve of 0.0022 decades would produce a negligible change in amplitude as compared with the tolerance limits, then V may be assumed equal to v_m without introducing a significant error. The replot of H_o for other values of v_m is not difficult; equation (52) provides for a quick calculation of corresponding values of V. The calculations should include small enough values of $\pm v_m$ so that between the smallest of these and zero H_o may be assumed to have zero slope within the prescribed tolerances.

The procedure now is almost the same as in the case of linear functions asymptotic at low frequencies, except that the procedure must be worked on both sheets of V. As noted above, the first element will have zero slope. The result of removing this element from the function of Figure 37 is shown on Figure 38.

The behavior of H_1 as V and V' take on large negative values may be analyzed as follows. Let

$$G_o(\omega^2) = G_o(\omega_m^2) + \frac{G_o^n}{n!}(\omega^2 - \omega_m^2)^n + \ldots\ldots$$

near ω_m, where n indicates the order of the first non-zero derivative at ω_m.

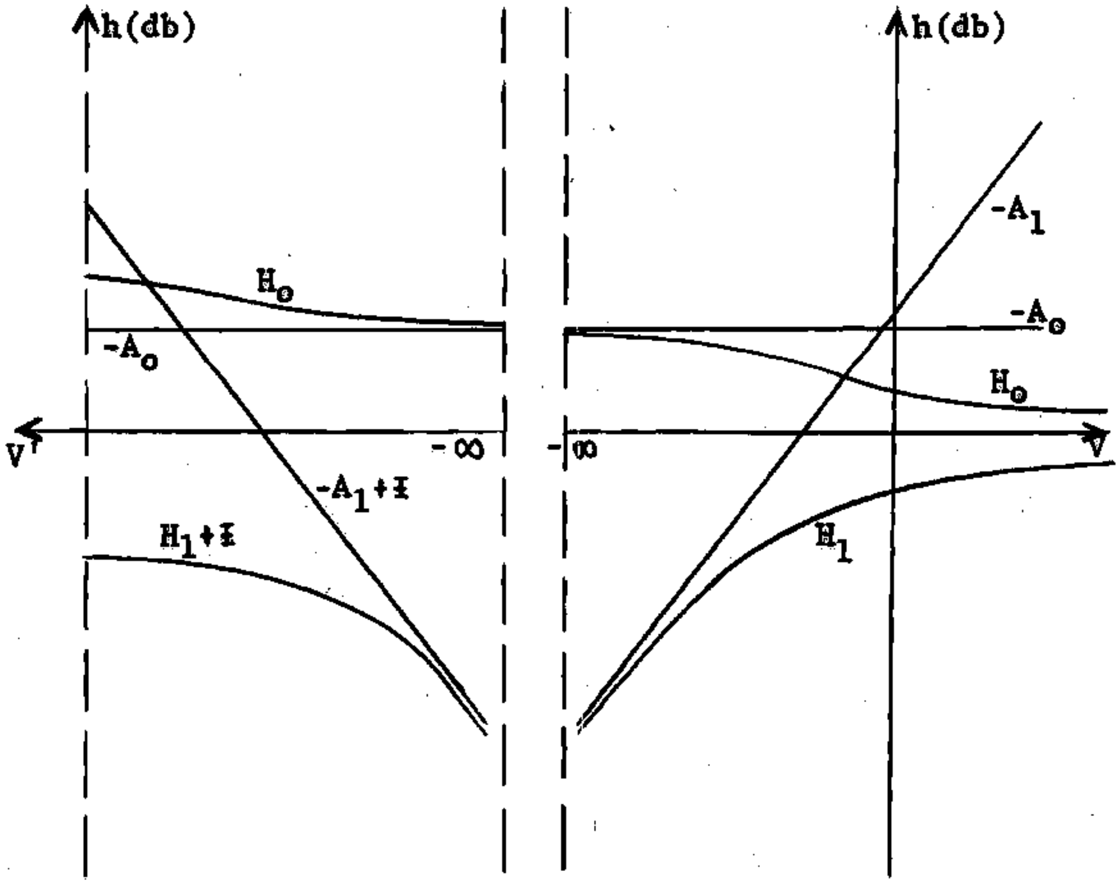

Fig. 38. First Remainder and Succeeding Linear Element with V as Abscissas.

$$G_1(\omega^2) = \frac{1}{G_o} - a_o \approx \frac{1}{G_o(\omega_m^2) + \frac{G_o^n}{n!}(\omega^2 - \omega_m^2)n} - \frac{1}{G_o(\omega_m^2)}$$

$$G_1(\omega^2) \approx \frac{-G_o^n(\omega^2 - \omega_m^2)^n}{n!\ (G_o(\omega_m^2))^2}$$

$$\frac{dH_1}{d\omega^2} = 10\frac{\frac{d}{d\omega^2}G_1(\omega^2)}{G_1(\omega^2)}(\log e) \approx \frac{10n \log e}{\omega^2 - \omega_m^2}$$

$$\frac{dV}{d\omega^2} = \frac{d}{d\omega^2}\left(\frac{1}{2}\log\left(\frac{\omega^2}{\omega_m^2} - 1\right)\right) = \frac{\log e}{2(\omega^2 - \omega_m^2)}$$

$$\frac{dV'}{d\omega^2} = \frac{d}{d\omega^2}\left(V + \frac{\mathrm{I}}{20}\right) = \frac{dV}{d\omega^2}$$

$$\frac{dH_1}{dV} = \frac{dH_1}{dV'} = \frac{dH_1}{d\omega^2}\cdot\frac{d\omega^2}{dV} = 20n \quad \text{for} \quad \omega \approx \omega_m$$

Thus the slope of H_1 will be an integral multiple of twenty decibels per decade. The method of linear asymptotes prescribes that $-A_1$ be chosen in the form

$$-A_1 = C + 20nV$$

For $\omega < \omega_m$, $V = V' + \frac{\mathrm{I}}{20}$

$$-A_1 = C + 20n\left(V' + \frac{\mathrm{I}}{20}\right) = C + 20nV' + n\mathrm{I}$$

Now if n is odd and C is real, $-A_1$ is complex for $\omega < \omega_m$. But if n is odd, $G_o(\omega^2)$ crosses $G_o(\omega_m^2)$ at ω_m; $G_o(\omega^2)$ may have a point of inflection at ω_m, but it does not have an extremum. Hence $G_1(\omega^2)$ changes sign at ω_m (as in the example of Figure 36, where n equals one) and $H_1(V')$ is also complex. On the other hand if n is even $G_o(\omega^2)$ has an extremum at ω_m, $G_1(\omega^2)$ does not change sign there, and $H_1(V)$ and $H_1(V')$ are both real or both complex. Since n‡ may be discarded if n is even, the method guarantees that $-A_1$ behaves in the same way as H_1 at the negative ends of both the V and the V' scales.

In the example of Figure 38 the proper $-A_1$ is shown. The method then proceeds exactly as in the case of expansion in terms of linear functions asymptotic at low frequencies, the only difference being that the problem is carried forward on two logarithmic sheets, one for real V and one for real V', instead of on one sheet. Example 5, Appendix A, gives a numerical illustration.

<u>Case where the prescribed function has a singularity at an intermediate frequency</u>.--In the first section of Chapter II it was pointed out that most typical prescribed magnitude functions do not have zeros at finite non-zero frequencies, but that a method would be advanced in this chapter to remove such zeros in those special cases where they appear. The method of the preceding section may be applied to this problem.

If G_o has a zero at ω_m, it can be represented by

$$G_o(\omega^2) = \frac{G_o^n}{n!}(\omega^2 - \omega^2)^n + \frac{G_o^{n+p}}{n+p!}(\omega^2 - \omega_m^2)^{n+p} + \dots .$$

where n, the order of the lowest non-zero derivative, must be even, and n+p is the order or the next non-zero derivative. This representation must be possible within the prescribed tolerance limits if a satisfactory rational function approximant exists, because such rational functions can be represented in this way. Choosing a_o to be the reciprocal of the first term, and solving equation (4) for G_1,

$$G_1(\omega^2) \approx - \frac{n!^2 G_o^{n+p}}{n+p!(G_o^n)^2}(\omega^2 - \omega_m^2)^{p-n}$$

The calculation of G_1 would actually be performed by replotting H_o as a function of V and applying the method of the preceding section. If p equals n, H_1 has zero slope. If not the next stage may be carried out against the scale of V as abscissas. Usually, however, the tolerance bounds will permit p to be chosen equal to n. Suppose, for instance that the usual constant tolerance limit in terms of decibels is prescribed. Then

$$G_o^{\pm} = \left(\frac{G_o^n}{n!} (\omega^2 - \omega_m^2)^n + \frac{G_o^{n+p}}{n+p!}(\omega^2 - \omega_m^2)^{n+p} - \ . \ . \ . \right) \varepsilon^{\pm 1}$$

where ε is a number usually very close to unity. Recalculating G_1,

$$G_1^{\mp} \approx - \frac{n!^2 G_o^{n+p}}{n+p!(G_o^n)^2}(\omega^2 - \omega_m^2)^{p-n} + \left(\frac{1 - \varepsilon^{\pm 1}}{\varepsilon^{\pm 1}}\right) \frac{1}{\frac{G_o^n}{n!}(\omega^2 - \omega_m^2)^n}$$

For $\omega^2 - \omega_m^2$ sufficiently small, evidently p can be chosen equal to n, and G_1 will still fall between the bounds imposed by the right hand term.

After the zero has been removed and a constant H_1 achieved, the remainder can be replotted to the original logarithmic variable v, and any of the other methods advanced herein can be pursued; or the solution may be carried out in terms of the modified variable V without reconversion.

A similar procedure may be used in the rare cases when G_o contains a double pole on the real ω axis.

Non-linear asymptotic elements.--In Chapter V the method treated involved expansion of the prescribed function in continued fraction form with linear elements asymptotic at infinity to corresponding remainders. In this chapter the expansion in low-frequency asymptotes again used linear functions. For the expansion with elements asymptotic at an intermediate frequency, the frequency scale was converted in such a way that linear functions could be used. It is not necessary in any of these methods for the elements to be restricted to the linear class.

In Figure 39 a high-frequency asymptotic function of the form

$$-A_k = \pm\ 10 \log b_k\left(1 + \frac{\omega^2}{\omega_k^2}\right) \tag{53}$$

is used. The advantage of such an element lies in the fact that it approximates the prescribed remainder over a greater range than does the

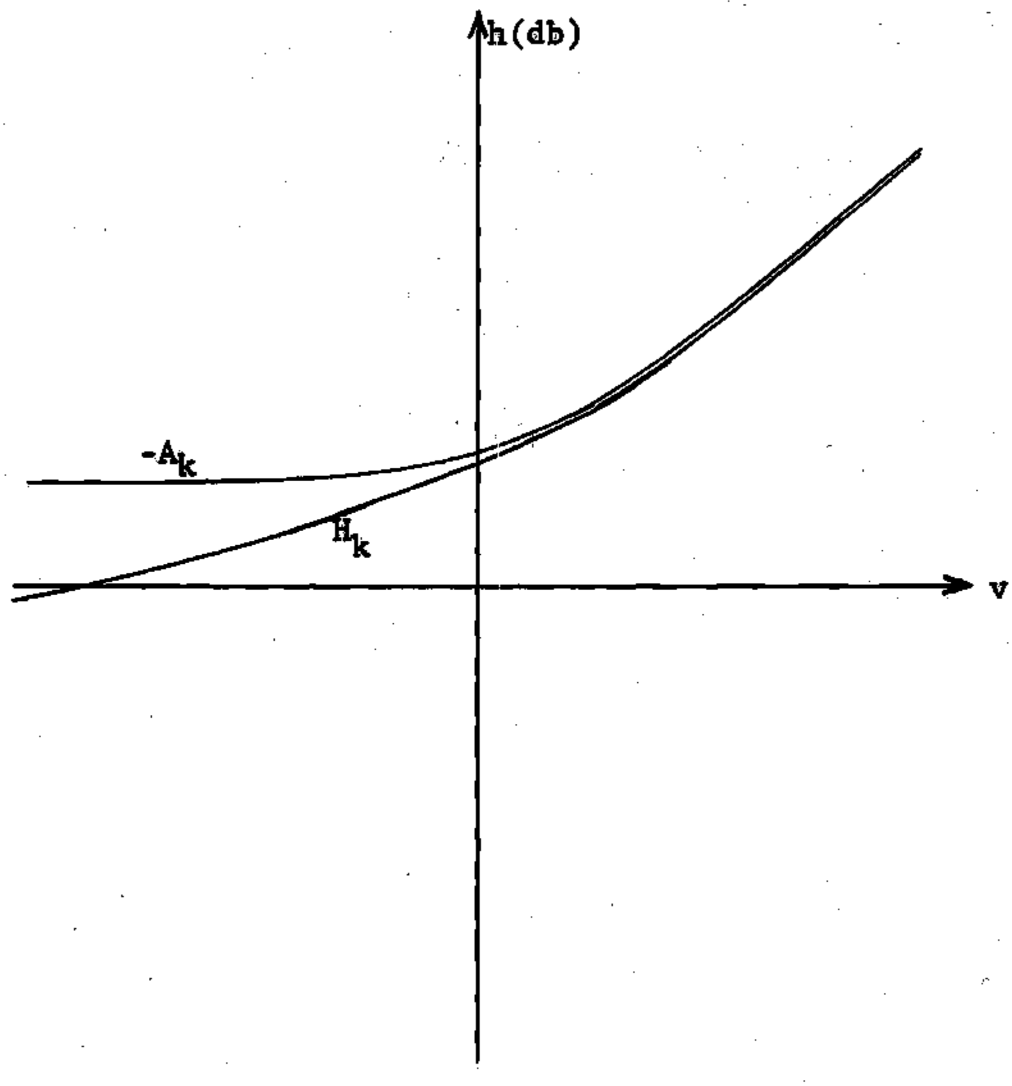

Fig. 39. Non-linear Asymptotic Element.

linear high-frequency asymptote, with the prospect that a successful final approximant may be achieved in fewer stages. Even better results might be anticipated if the class of available functions were extended to include the form

$$-A_k = \pm 10 \log b_k(1 + 2c_k\frac{\omega^2}{\omega_k^2} + \frac{\omega^4}{\omega_k^4}) \tag{54}$$

An inherent drawback is that selection of non-linear $-A_k$ is more difficult numerically than the selection of linear ones. The former are tabulated in Appendix B with respect to a normalized origin, (b_k, ω_k) equals (1,1), but only at intervals of 0.1 of log ω. If in fitting a proposed element to the given curve, it develops that ω_k does not happen to be removed from unity by an integral multiple of one tenth of a decade, the tabulated values of either the given curve or the proposed element cannot be used, and additional point by point values of one or the other must be calculated. This can be achieved by use of the L(u) tables, as shown in the next section. It is not a burdensome task numerically if the series of calculations only has to be performed once; but where an element is being fitted asymptotically to a curve, often several trials have to be made to determine the best choice. With $-A_k$ of the form of equation (54) an additional choice must be made for the coefficient $2c_k$. Curves of this type are tabulated in Appendix B for integral values of $10 \log(2c_k + 2)$, which are the values of the normalized functions for ω equal to one, but, of course, a selection different from the tabulated ones might be required. From the point of view of simplicity of procedure it appears that the selection of non-linear elements may be advantageously utilized in the circumstance that one of the tabulated standard components happens to fit the requirements for the element, but that in other cases the amount of computation involved makes the use of linear elements preferable.

Elements asymptotic at zero and infinity.--One case in which it is feasible to select non-linear asymptotic elements occurs if each element is designed to approach the high-frequency asymptote of the corresponding remainder as ω increases without bound and to approach the low-frequency asymptote as ω approaches zero. The selection of such a function is simplified because its constants are determined by the location of the two asymptotes, and as a result only one series of calculations of the points on the element function need be made.

The selection of $-A_k$ in this method is made as follows. Let $-A_k^o$ be the asymptote of H_k for small ω and $-A_k^i$ be the asymptote for large ω.

$$-A_k^o = 10 \log b_k^o \, \omega^{2p_k^o}$$

$$-A_k^i = 10 \log b_k^i \, \omega^{2p_k^i}$$

A suitable non-linear $-A_k$ must approach $-A_k^o$ for small ω and $-A_k^i$ for large ω. A simple choice for $-A_k$ which meets these requirements is:

$$-A_k = 10 \log \frac{1}{\frac{1}{b_k^o \omega^{2p_k^o}} + \frac{1}{b_k^i \omega^{2p_k^i}}} \qquad \text{for } p_k^o > p_k^i \tag{55}$$

$$-A_k = 10 \log(b_k^o \omega^{2p_k^o} + b_k^i \omega^{2p_k^i}) \qquad \text{for } p_k^i > p_k^o \tag{56}$$

Other choices are possible. In effect we have selected here the choice which has the lowest order with which the asymptotic requirements can be met and which approaches these asymptotes in the Taylor sense, that is, the difference between $-A_k$ and the asymptotes is maximally flat at zero and infinity.

The formulae for calculating $-A_k$ point by point are:

$$-A_k = -A_k^o - L(I - A_k^o + A_k^i) \qquad \text{for } p_k^o > p_k^i \tag{57}$$

$$-A_k = -A_k^o + L(I + A_k^o - A_k^i) \qquad \text{for } p_k^i > p_k^o \tag{58}$$

Neither of the above formulae will apply to the case where p_k^o equals p_k^i. This special case occurs when the high-frequency and low-frequency asymptotes are parallel.

In all respects other than those set forth above the procedure is the same as in cases of linear elements. It will even work in cases where $-A_k^o$ is real and $-A_k^i$ is complex, or vice-versa (again with the exception of the parallel asymptotes case), but may not be advisable if the singularities of H_k and the element do not coincide. Example 6, Appendix A, illustrates an application of the method where $-A_k$ is asymptotic to H_k at both high and low frequencies.

CHAPTER VII

ACCURACY OF THE METHOD

Sources of errors.--The accuracy of the semi-graphical method of approximation described in the preceding chapters is important not only as an indication of its areas of usefulness but also as a guide in applying the method itself. The accuracy depends on the choices made for the various A_k as well as on inherent properties of the method. It is also dependent, of course, on the number of decimal places to which the tables are calculated.

The tables of standard component functions (Appendix B) and the tables of L(u) (Appendix C) have been carried out to the nearest thousandth of a decibel, but are designed principally for calculations to be made to the nearest hundredth of a decibel, as entry into the tables with values to the nearest thousandth cannot be made without tedious interpolation. Interpolation to gain greater accuracy is discussed in Appendix C. All the operations involved in the method are entries into one of the tables or additions or subtractions. The principal recurrence formulae are equation (12) for calculating successive remainders and equation (13) for calculating the approximants to remainders and the approximant to the prescribed function. Any error in the result of any operation indicated by those equations will be expressed in decibels. The error in addition operations is merely the sum of the errors in the quantities added. The error in a figure obtained from the tables and rounded off to the nearest hundredth of a decibel is 0.005 db. provided the quantity with which entry was made into the tables is exact. An error of 0.005 db. in H_k corresponds to an error of about 0.115 per cent in G_k.

Errors introduced by the tables of L(u).--Let N_o^* be the true value of a quantity for which the calculated value is N_o. Let the limits of the error in N_o be expressed by the equation

$$N_o + e_o^- \leqq N_o^* \leqq N_o + e_o^+$$

Let $N_1^* = L(N_o^*)$ and let N_1 be the calculated value of N_1^* obtained by entering the tables with N_o. The error limits associated with N_1 are

$$N_1 + e_1^- \leqq N_1^* \leqq N_1 + e_1^+$$

Before proceeding further we must make clear what is meant by the inequality signs above as related to quantities which may be involved in entering the L(u) tables. To do this we draw the curve of L(u) versus u in a particular way as shown in Figure 40. Note the choice of axes, particularly the reversed scales on the $u + I$ and $L(u) + I$ axes, which make the points $-\infty$ and $-\infty + I$ coincide. The points $+\infty$ and $\infty + I$ are also regarded as contiguous so that L(u) may be treated as a continuous curve. Inequality is interpreted in the sense of the figure; that is, u_1 greater than u_2 implies that u_1 is

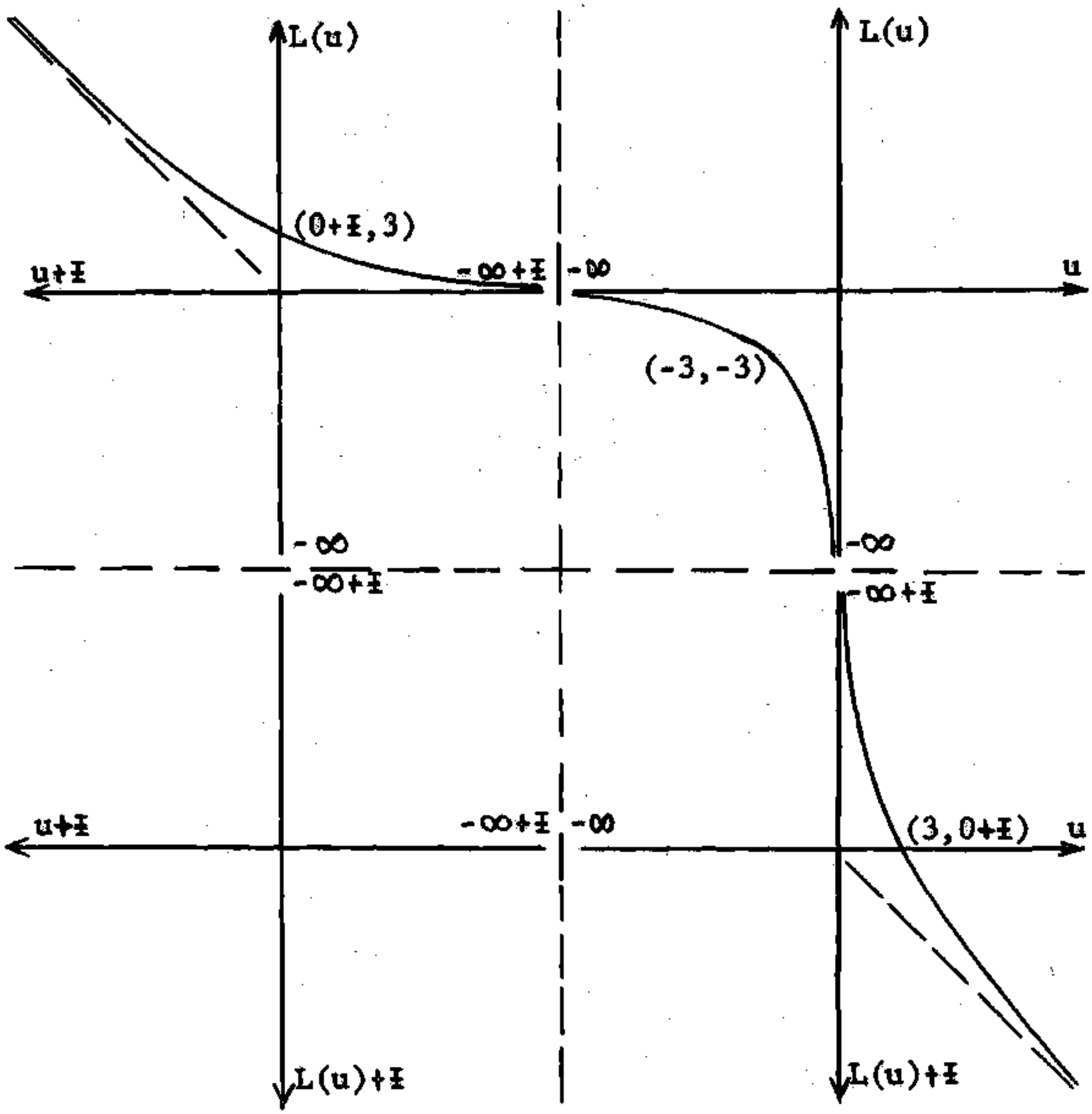

Fig. 40. Sketch of L(u).

to the right of u_2, and $L(u_1)$ greater than $L(u_2)$ implies $L(u_1)$ is above $L(u_2)$, in Figure 40. As a result of this definition the following algebraic relations hold:

$$a > b \quad \text{implies} \quad a + c > b + c \qquad (c \text{ real})$$

$$a > b \quad \text{implies} \quad a + c + I < b + c + I \qquad (c \text{ real})$$

$$a > b \quad \text{implies} \quad L(a) < L(b)$$

Returning to the error limits for N_o, application of the third relation above yields

$$L(N_o + e_o^-) \geqq N_1^* \geqq L(N_o + e_o^+)$$

One of the algebraic identities of $L(u)$ functions, developed in Appendix C, which is frequently used in this chapter, is

$$L(u + w) = L(u) + L\left\{L(w) - L(-u)\right\} \tag{59}$$

From this equation

$$L(N_o) + L\left\{L(e_o^-) - L(-N_o)\right\} \geqq N_1^* \geqq L(N_o) + $$

$$+ L\left\{L(e_o^+) - L(-N_o)\right\}$$

If the tables of $L(u)$ were calculated to a large number of decimal places, we could write

$$L(N_o) = N_1, \text{ and therefore}$$

$$e_1^{\pm} = L\left\{L(e_o^{\mp}) - L(-N_o)\right\} \tag{60}$$

Since the tables are rounded off at a limited number of decimal places, an additional component of error must be added as shown in the next section. Thus equations (60) give only the contribution to the error in N_1 that arises from the error in N_o.

Errors in the calculation of remainders.--Consider one form of equation (12),

$$H_{k+1} = -H_k + L(A_k + H_k) \tag{61}$$

which is used to calculate successive remainders. Let e_k^+ and e_k^- be the limits of error for H_k, $e_{k+1}^{\pm}$ the corresponding limits for H_{k+1}, and d_{k+1} the component of error in H_{k+1} introduced by rounding off the quantities obtained from the tables of $L(u)$.

First, a distinction should be made between non-linear and linear A_k. For non-linear A_k the values for each point are obtained from the tabulated standard components of Appendix B or the $L(u)$ tables of Appendix C. Thus for non-linear A_k a component of error must be included due to a possible error in A_k itself. On the other hand, linear A_k are chosen directly by observation of the logarithmic remainders. They have the form $C_k + 20nv$. If, for example, C_k is chosen to be 7.23, and the intervals of v are chosen so that 20nv has no decimal part less than 0.01, then $A_k = 7.23 + 20nv$ is exact for all v, and no error is introduced with A_k. In the examples of Appendix A, intervals of v of 0.1 are used, but even if intervals of 0.001 were used the linear A_k would be exact. Let the error associated with A_k, if any, be $\pm d_{ak}$.

The lower error limit for H_{k+1} is calculated as follows.

$$H_{k+1} + e^-_{k+1} = - H_k - e^+_k + L(A_k + d_{ak} + H_k + e^+_k) - d_{k+1}$$

Making use of equation (60),

$$e^-_{k+1} = - e^+_k - d_{k+1} + L\left\{L(d_{ak} + e^+_k) - L(-A_k - H_k)\right\}$$

From equation (59) and other identities of the algebra of L(u)

$$e^-_{k+1} = - e^+_k + e^+_k + d_{ak} - d_{k+1} + L\left\{L\left[-L(-A_k - H_k)\right]\right.$$

$$\left. - L\left[-L(e^+_k + d_{ak})\right]\right\}$$

$$= d_{ak} - d_{k+1} + L\left\{L(-e^+_k - d_{ak}) - L(A_k + H_k)\right\}$$

The same result is obtained if the alternative form of equation (61), given in equation (12), is used. The other error limit is obtained in a similar manner, and both may be combined to yield

$$e^{\pm}_{k+1} = \mp d_{ak} \pm d_{k+1} + L\left\{L(-e^{\mp}_k \pm d_{ak}) - L(A_k + H_k)\right\} \qquad (62)$$

If linear A_k are used, the equations become

$$e^{\pm}_{k+1} = \pm d_{k+1} + L\left\{L(-e^{\mp}_k) - L(A_k + H_k)\right\} \qquad (63)$$

A note on the value of d_{k+1} must be interpolated here. The magnitude of d_{k+1} depends on the number of decimal places to which the quantity extracted from the tables is carried; where values are rounded off to two decimal places, d_{k+1} equals 0.005. Since the superscripts on $e^{\pm}_{k+1}$ identify 'upper' and 'lower' limits in the sense of Figure 40 as explained above, it follows that if H_{k+1} is real, d_{k+1} is positive, and if H_{k+1} is complex, d_{k+1} is negative. The subscript on d_{k+1} is retained to associate it with H_{k+1} so that the proper sign can be ascribed to it in the final result.

The recurrence formulae (62) and (63), though precise, lead to very cumbersome results in an expression for the error associated with the last remainder. To obtain a more useful approximate expression for error we make the following assumptions.

(1) Where e_k is the sum of several components, the resulting error in e_{k+1} is taken as the sum of the results obtained by applying the L-operation error formula separately to the individual components. Thus when

$$e^{+}_{k} = \sum_{p} {}_{p}e^{+}_{k},$$

$$L\left\{L(e^{+}_{k}) - L(A_k + H_k)\right\} = \sum_{p} L\left\{L({}_{p}e^{+}_{k}) - L(A_k + H_k)\right\}$$

The effect of the assumption is to base the error projection more nearly on the slope and curvature of L(u) at $A_k + H_k$ rather than considering how these quantities change as L(u) moves away from $L(A_k + H_k)$.

(2) For the second assumption

$${}_{p}e^{+}_{k} = -\,{}_{p}e^{-}_{k}$$

This assumption is very good for small ${}_pe_k$; for ${}_pe_k$ equal to 0.01 db., the usual value encountered, the error in the assumption is less than 0.005 db. unless the difference between H_k and $-A_k$ is less than three-tenths of a decibel, and in the latter region the spread in the tolerance bounds will be so much greater than the error in exchanging error limits that the error in this assumption is not important.

We may now proceed to develop an approximate expression for the errors associated with successive remainders, considering first the case of linear A_k. Let the error associated with H_o be $\pm d_o$ for symmetry of notation. Then

$$- e_o^- = -(-d_o) = d_o$$

$$e_1^+ = d_1 + L\left\{L(d_o) - L(A_o + H_o)\right\}$$

$$\begin{aligned} e_2^+ &= d_2 + L\left\{L(-e_1^-) - L(A_1 + H_1)\right\} \\ &= d_2 + L\left\{L(e_1^+) - L(A_1 + H_1)\right\} \\ &= d_2 + L\left\{L\left[d_1 + L(L(d_o) - L(A_o + H_o))\right] - L(A_1 + H_1)\right\} \\ &= d_2 + L\left\{L(d_1) - L(A_1 + H_1)\right\} + \\ &\qquad + L\left\{L\left[L(L(d_o) - L(A_o - H_o))\right] - L(A_1 + H_1)\right\} \\ &= d_2 + L\left\{L(d_1) - L(A_1 + H_1)\right\} + L\left\{L(d_o)\right. \\ &\qquad \left. - L(A_o + H_o) - L(A_1 + H_1)\right\} \end{aligned}$$

Extending the same procedure to successive stages for both upper and lower limits,

$$e_n^{\pm} = \pm d_n + \sum_{k=0}^{k=n-1} L\left\{L(\pm d_k) - \sum_{p=k}^{p=n-1} L(A_p + H_p)\right\} \tag{64}$$

For the case of non-linear A_k the additional term below must be added to the above expression to obtain the error in $e_n^{\pm}$.

$$\sum_{k=0}^{k=n-1} L\left\{L(\mp d_{ak}) - L(-A_k - H_k) - \sum_{p=k+1}^{p=n-1} L(A_p + H_p)\right\}$$

<u>Error in the final approximant</u>.--We now come to the inverse process. The equation for calculating approximants is

$$H_{kn} = -A_k - L(-A_k + \bar{I} + H_{k+1,n}) \tag{65}$$

The first step in the successive calculations is

$$H_{n-1,n} = -A_{n-1} - L(-A_{n-1} + \bar{I} + H_{nn})$$

where $H_{nn} = -A_n$, the last term in the expansion. These calculations are not actually performed in the method, except as a check when desired. Instead, when the last A_n has been determined, G_{on} is formed as follows:

$$G_{on} = \cfrac{1}{10^{\frac{A_o}{10}} + \cfrac{1}{10^{\frac{A_1}{10}} + \cdots + \cfrac{1}{10^{\frac{A_n}{10}}}}} \tag{66}$$

The antilogarithms of the various $A_k/10$ are taken, and the result is reduced to the form of a rational fraction. Any errors introduced in this part of the procedure stem from inadequacy in the tables of logarithms used in calculating the $a_k(\omega^2)$, and have nothing to do with the $L(u)$ tables. We assume herein that the tables of logarithms used are adequate so that the difference between G_{on} and its representation in the form of equation (66) is negligible; in fact, equation (66) is the solution from our viewpoint. This amounts to stating that in the sequence of calculations (65) no errors are present due to the use of tables.

The last element in the expansion, $-A_n$, is ideally chosen equal to H_n. Any difference is due to the fact that H_o is being approximated rather than exactly reconstructed, and is not due to the introduction of errors, which is the topic being investigated here. Letting H_n^* represent the true value of H_n,

$$H_{nn} + e_{nn}^- \leqq H_n^* \leqq H_{nn} + e_{nn}^+$$

where $e_{nn}^{\pm}$ are the limits of error between the approximant to the last remainder and the true value of the last remainder. Similarly $e_{kn}^{\pm}$ will denote the limits of error between the approximant to the k^{th} remainder and the true value of the k^{th} remainder. Choosing for H_{nn} its ideal value H_n, it follows that $e_{nn}^{\pm} = e_n^{\pm}$. We wish to determine $e_{on}^{\pm}$.

From equation (65) the following recurrence formula is derived in the same manner as in the preceding section.

$$e_{kn}^{\pm} = - L\left\{L(e_{k+1,n}^{\mp}) - L(A_k + I - H_{k+1,n})\right\}$$

Also from equation (65),

$$A_k + H_{kn} = -L(-A_k + I + H_{k+1,n})$$

$$L(A_k + H_{kn}) = L\left\{-L(-A_k + I + H_{k+1,n})\right\}$$

$$= -L(A_k + I - H_{k+1,n})$$

Making the indicated substitution, and also using the assumption, $e^{\pm}_{kn} = -e^{\mp}_{kn}$, we find that the recurrence formula becomes

$$e^{\pm}_{kn} = L\left\{L(e^{\pm}_{k+1,n}) + L(A_k + H_{kn})\right\}$$

which leads by successive application to

$$e^{\pm}_{on} = L\left\{L(e^{\pm}_{nn} + \sum_{k=0}^{k=n-1} L(A_k + H_{kn})\right\}$$

If H_{nn} is replaced by H_n, e_{nn} becomes e_n as indicated above, and all H_{kn} become H_k, so that

$$e^{\pm}_{on} = L\left\{L\left[\pm d_n + \sum_{k=0}^{k=n-1} L(L(\pm d_k) - \sum_{p=k}^{p=n-1} L(A_p + H_p))\right] + \sum_{k=0}^{k=n-1} L(A_k + H_k)\right\}$$

The expression in brackets above is the sum of n+1 components. By the use of the assumption,

$$L\left\{L\left(\sum_p {}_pe_k\right) - M\right\} \approx \sum_p L\left\{L({}_pe_k) - M\right\},$$

which was introduced in the preceding section, and some algebraic manipulation, we obtain an alternative expression for $e_{on}^{\pm}$, which, because of some cancellations, is considerably simpler.

$$e_{on}^{\pm} = \sum_{k=0}^{k=n} L\left\{L(\pm d_k) + \sum_{p=0}^{p=k-1} L(A_p + H_p)\right\} \tag{67}$$

The term to be added for the case of non-linear A_k is

$$\sum_{k=0}^{k=n-1} L\left\{L(\mp d_{ak}) + A_k + H_k + I + \sum_{p=0}^{p=k-1} (L(A_p + H_p)\right\}$$

<u>Implications for the selection of A_k</u>.--We note from equation (67) for $e_{on}^{\pm}$ that the latter is expressed as the sum of terms of the form

$${}_ke_{on}^{\pm} = L\left\{L(\pm d_k) + M_k\right\}$$

where $M_k = \sum_{p=0}^{p=k-1} L(A_p + H_p)$. Table 2 illustrates the effect of M_k on ${}_ke_{on}^{\pm}$ for d_k equal to 0.01 db.

Table 2. Relation Between Component of Error and M_k.

M_k	${}_ke^{+}_{on}$	${}_ke^{-}_{on}$
- 3.01	0.005	-0.005
0	0.010	-0.010
1.76	0.015	-0.015
3.01	0.020	-0.020
10.00	0.099	-0.101
15.00	0.306	-0.328
20.00	0.901	-1.136
25.00	2.379	-5.651
26.38	3.015	- ∞

From

$$M_k = \sum_{p=0}^{p=k-1} L(A_p + H_p)$$

$$= \sum_{p=0}^{p=k-1} (H_{p+1} + H_p)$$

$$= H_o + \sum_{p=1}^{p=k-1} (2H_p) + H_k$$

and since H_o is real and $2H_p$ must be real whether H_p is or not, M_k is complex only if H_k is complex. But if H_k is complex, $+d_k$ is negative, $L(+d_k)$ is real, and $L(+d_k) + M_k$ is complex, just as it was for real M_k. Thus ${}_ke_{on}^{\pm}$ in Table 2 have almost exactly the same values if the left column represents the real part of a complex M_k. The table shows that to maintain accuracy the real part of M_k, and hence the real part of each $L(A_k + H_k)$, should be kept small. If every $L(A_k + H_k)$ has a negative real part, then sums of the type M_k will have even more negative real parts, and the overall error will be less than 0.01 db. per element in the expansion. This desirable result occurs if $A_k + H_k$ is real and less than three, that is, if $-A_k$ falls above a point three decibels below H_k at every stage of the expansion.

<u>Use of error equations</u>.--The error obtained in equation (67) is applied to H_o to give the limits $H_o + e_{on}^{+}$ and $H_o + e_{on}^{-}$, between which H_{on}, the final approximant, will fall. If these error limits fall within the

prescribed tolerances, H_{on} is a satisfactory approximant to H_o. However, it must be recalled that equation (67) was based on the selection of $-A_n$ equal to H_n, the last remainder. As this is not generally the case, it will usually not be possible to use equation (67) in this manner.

Generally what is wanted is a criterion that will tell, when $-A_n$ is selected, that the final result will be acceptable. One way to accomplish this is to use equation (64) to calculate a lower error limit for the upper tolerance bound H_n^+, and an upper error limit for the lower tolerance bound H_n^-. The values of $L(A_k + H_k^+)$ and $L(A_k + H_k^-)$, necessary for these calculations, will have already been obtained in the course of the expansion, so that the amount of additional computation involved to check a few critical points is not excessive.

Probably the simplest method is to incorporate a compensation for possible errors into the upper and lower tolerance bounds at each stage of the expansion. This may be accomplished by shifting each bound 0.01 db. toward the other bound after they have been calculated in the usual way from the bounds of the preceding stage, thereby narrowing slightly the band between the two bounds. Then if $-A_n$ falls between the modified H_n tolerance bounds for all frequencies, the approximant obtained by terminating the expansion at that stage will surely be acceptable. This procedure is well on the cautious side, as the shift, although the smallest that can be made, is twice as great as the one theoretically necessary to compensate for maximum error in the case of linear A_k.

APPENDIX A

NUMERICAL EXAMPLES

<u>First example: mechanics of the logarithmic calculations.</u>--Figure 41 depicts the graph of a prescribed function, H_o, plotted against the logarithmic frequency scale, $v = \log \omega$. The function is tabulated at intervals of one-tenth of a decade in column (2) of Table 3 below. Column (3) gives the values of $-A_o$, selected in accordance with the particular method or strategy being used. In this case $-A_o$ is $-10 \log(1 + \omega^2)$; its values may be obtained from Appendix B. Succeeding columns give the intermediate values obtained in the course of calculating H_1:

Column (4): $A_o + H_o$

Column (5): $L(A_o + H_o)$, obtained from the tables in Appendix C.

Column (6): H_1, from $H_1 = - H_o + L(A_o + H_o)$

H_1 is also shown plotted in Figure 41. It is labeled $H_1 + I$ in the figure because, as indicated in column (6), the values of H_1 are complex and only the real part can be plotted. This completes a typical step in the expansion procedure. The next step would be to choose a function for $-A_1$, and repeat the procedure to calculate H_2.

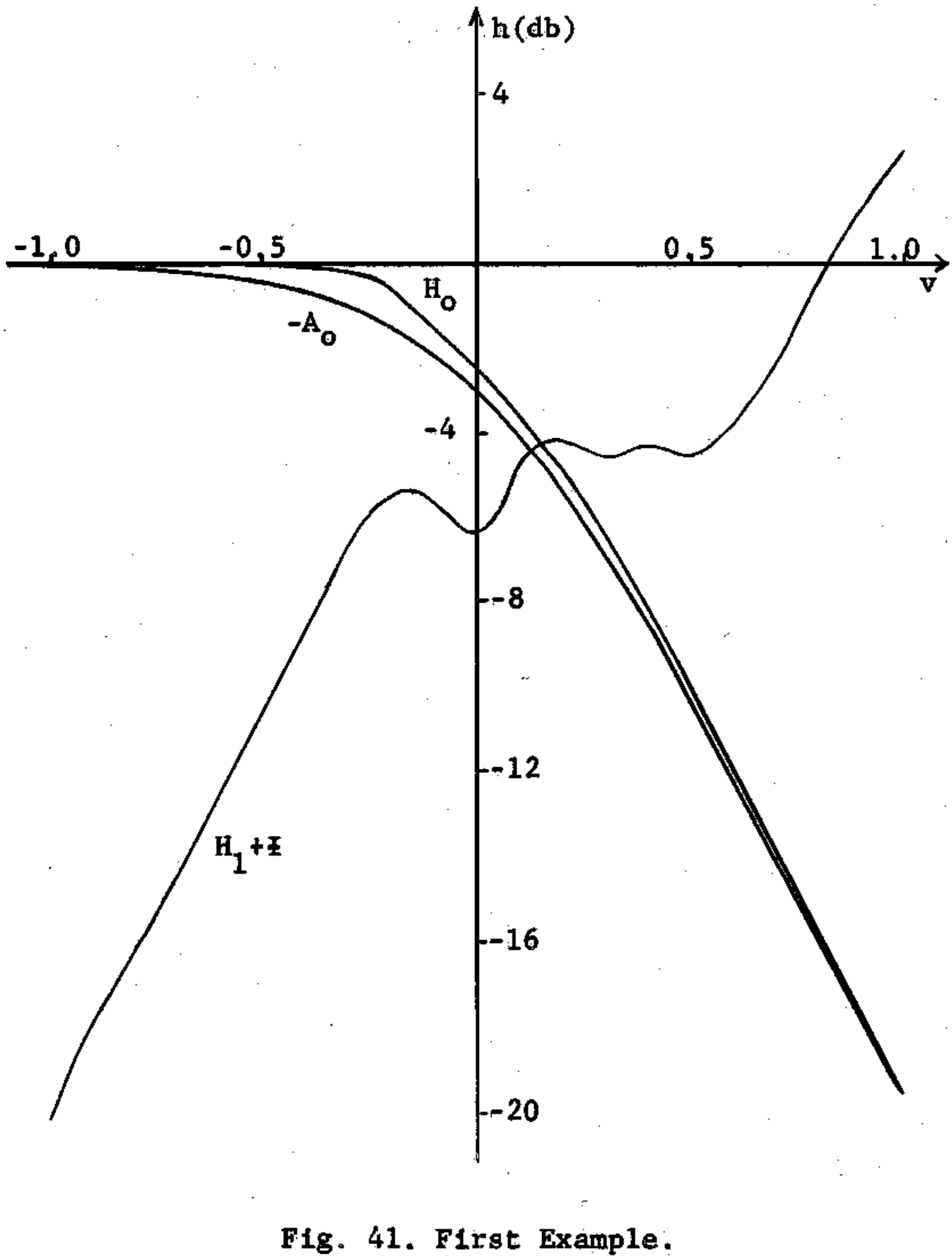

Fig. 41. First Example.

Table 3. Values of Quantities Pertaining to First Example

The column headings in parentheses identify the several columns for reference in the accompanying text, where the columns are explained.

v (1)	H_o (2)	$-A_o$ (3)	(4)	(5)	H_1 (6)
-1.00	0	- 0.04	0.04	-20.34+I	-20.34+I
-0.90	0	- 0.07	0.07	-17.89+I	-17.89+I
-0.80	- 0.01	- 0.11	0.10	-16.33+I	-16.32+I
-0.70	- 0.02	- 0.17	0.15	-14.54+I	-14.52+I
-0.60	- 0.03	- 0.27	0.24	-12.46+I	-12.43+I
-0.50	- 0.05	- 0.41	0.36	-10.63+I	-10.58+I
-0.40	- 0.09	- 0.64	0.55	- 8.70+I	- 8.61+I
-0.30	- 0.17	- 0.97	0.80	- 6.94+I	- 6.77+I
-0.20	- 0.50	- 1.46	0.96	- 6.07+I	- 5.57+I
-0.10	- 1.50	- 2.12	0.62	- 8.14+I	- 5.64+I
0	- 2.50	- 3.01	0.51	- 9.04+I	- 6.54+I
0.10	- 3.50	- 4.12	0.62	- 8.14+I	- 4.64+I
0.20	- 4.97	- 5.46	0.49	- 9.23+I	- 4.26+I
0.30	- 6.64	- 6.97	0.31	-11.31+I	- 4.67+I
0.40	- 8.41	- 8.64	0.23	-12.65+I	- 4.24+I
0.50	-10.27	-10.41	0.14	-14.85+I	- 4.58+I
0.60	-12.17	-12.27	0.11	-15.91+I	- 3.74+I
0.70	-14.06	-14.17	0.10	-16.33+I	- 2.27+I
0.80	-16.01	-16.11	0.10	-16.33+I	- 0.32+I
0.90	-17.98	-18.07	0.09	-16.79+I	1.19+I
1.00	-19.96	-20.04	0.08	-17.31+I	2.65+I

<u>Second example: approximation with general rational functions</u>.--In this example $-A_k$ are chosen less than the corresponding H_k. Instead of carrying forward both tolerance bounds, only the remainders of the prescribed function are developed, reliance being placed on some of the relations of Chapter III to determine when a satisfactory approximant has been reached.

The prescribed function is depicted in Figure 42. The approximant desired is to be equal to or less than H_o, but not more than 0.50 db. less than H_o. For frequencies less than those represented in Figure 42, H_o is

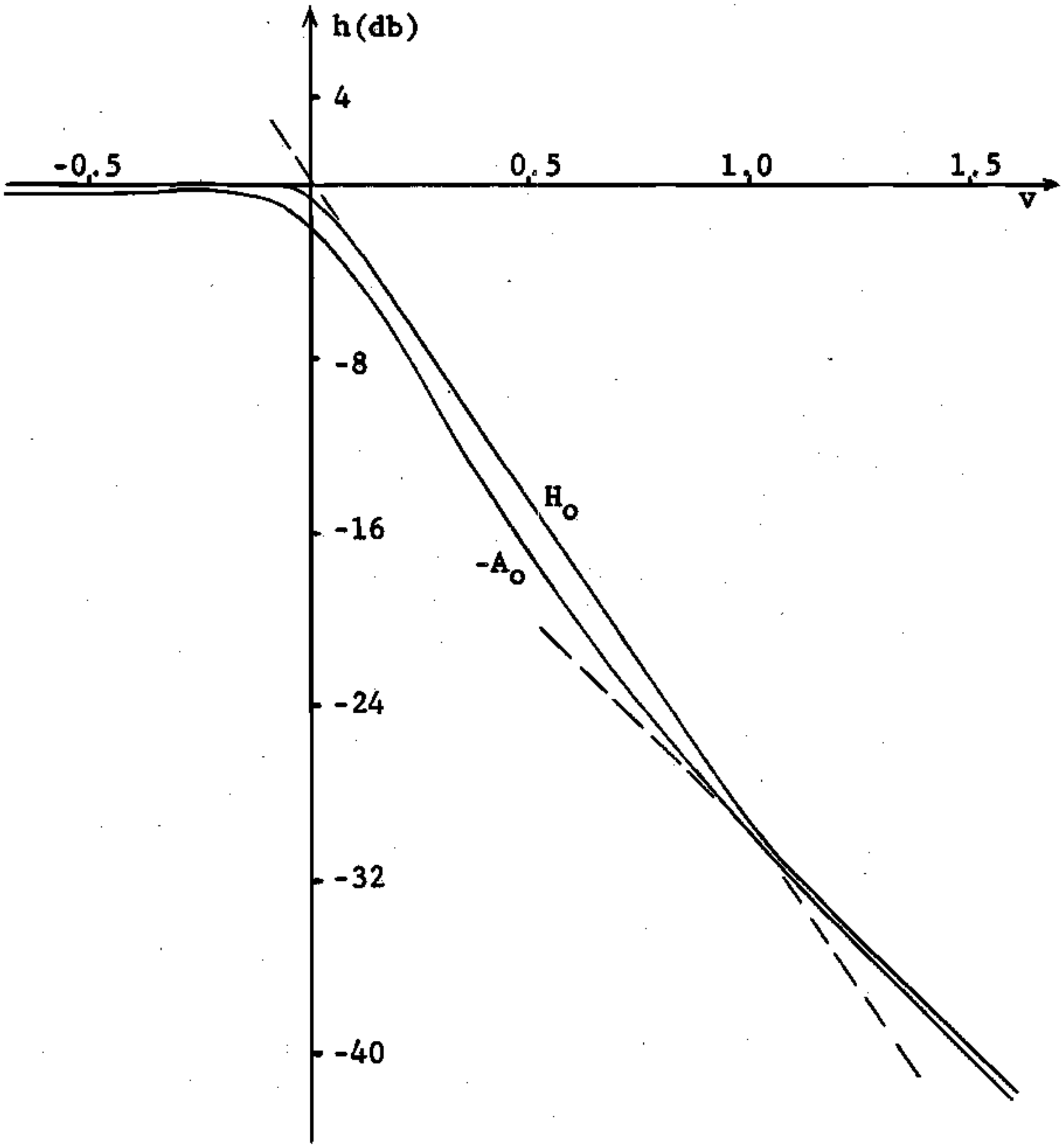

Fig. 42. Second Example.

assumed equal to zero, and for those greater than the indicated range, H_o is assumed to continue to drop off at twenty decibels per decade. The appropriate columns of quantities involved in solving this problem are given in Table 4, following which the steps in the calculation are discussed.

Table 4. Values of Quantities Pertaining to Second Example

The column headings in parentheses identify the several columns for reference in the accompanying text, where the columns are explained.

(1)	(2)	(3)	(4)	(5)	(6)	(7) ±
-1.0	0	- 0.48	0	- 0.48	0.48	- 9.32
-0.9	0	- 0.47	0.01	- 0.46	0.46	- 9.52
-0.8	0	- 0.46	0.01	- 0.45	0.45	- 9.62
-0.7	0	- 0.43	0.02	- 0.41	0.41	-10.04
-0.6	0	- 0.40	0.03	- 0.37	0.37	-10.51
-0.5	0	- 0.36	0.04	- 0.32	0.32	-11.17
-0.4	0	- 0.32	0.07	- 0.25	0.25	-12.27
-0.3	0	- 0.32	0.11	- 0.21	0.21	-13.05
-0.2	0	- 0.47	0.17	- 0.30	0.30	-11.46
-0.1	0	- 1.06	0.27	- 0.79	0.79	- 7.00
0	- 0.75	- 2.50	0.41	- 2.09	1.34	- 3.67
0.1	- 3.00	- 5.06	0.64	- 4.42	1.42	- 1.13
0.2	- 6.00	- 8.47	0.97	- 7.50	1.50	2.15
0.3	- 9.00	-12.32	1.46	-10.86	1.86	6.28
0.4	-12.00	-16.32	2.12	-14.20	2.20	10.19
0.5	-15.00	-20.36	3.01	-17.35	2.35	13.56
0.6	-18.00	-24.20	4.12	-20.28	2.28	16.39
0.7	-21.00	-28.44	5.46	-22.98	1.98	18.62
0.8	-24.00	-32.46	6.97	-25.49	1.49	20.12
0.9	-27.00	-36.47	8.64	-27.83	0.83	20.23
1.0	-29.75	-40.48	10.41	-30.07	0.32	18.58
1.1	-32.00	-44.49	12.27	-32.22	0.22	19.16
1.2	-34.00	-48.49	14.17	-34.32	0.32	22.83
1.3	-36.00	-52.50	16.11	-36.39	0.39	25.73
1.4	-38.00	-56.50	18.07	-38.43	0.43	28.17
1.5	-40.00	-60.50	20.04	-40.46	0.46	30.48
1.6	-42.00	-64.50	22.03	-42.47	0.47	32.58
1.7	-44.00	-68.50	24.02	-44.48	0.48	34.68
1.8	-46.00	-72.50	26.01	-46.49	0.49	36.77
1.9	-48.00	-76.50	28.01	-48.49	0.49	38.77
2.0	-50.00	-80.50	30.00	-50.50	0.50	40.86

Table 4. Values of Quantities Pertaining to Second Example
(continued)

(1)	(8)	(9)	(10) ±	(11)	(12)	(13)
-1.0	-47.00	16.11	-30.89	21.57	30.86	
-0.9	-43.00	14.17	-28.83	19.31	28.78	
-0.8	-39.00	12.27	-26.73	17.11	26.64	
-0.7	-34.99	10.41	-24.58	14.54	24.42	
-0.6	-30.99	8.64	-22.35	11.84	22.06	
-0.5	-26.98	6.97	-20.01	8.84	19.40	
-0.4	-22.97	5.46	-17.51	5.24	15.97	
-0.3	-18.95	4.12	-14.83	1.78	10.10	
-0.2	-14.92	3.01	-11.91	0.45	1.84	
-0.1	-10.88	2.12	- 8.76	1.76	3.99	
0	- 6.81	1.46	- 5.35	1.68	0.41	
0.1	- 2.71	0.97	- 1.74	0.61	- 7.09	
0.2	1.42	0.64	2.06	0.09	-18.94	
0.3	5.57	0.41	5.98	0.30	-17.74	
0.4	9.64	0.27	9.91	0.28	-21.96	
0.5	13.31	0.17	13.48	0.08	-30.87	
0.6	16.00	0.11	16.11	0.28	-28.16	-32.00
0.7	17.31	0.07	17.38	1.24	-23.43	-26.04
0.8	17.64	0.04	17.68	2.44	-21.35	-21.35
0.9	17.57	0.03	17.60	2.63	-21.03	-22.04
1.0	17.42	0.02	17.44	1.14	-23.81	-24.00
1.1	17.29	0.01	17.30	1.86	-21.98	-25.30
1.2	17.19	0.01	17.20	5.63	-18.59	
1.3	17.12	0	17.12	8.61	-17.76	
1.4	17.08	0	17.08	11.09	-17.33	
1.5	17.05	0	17.05	13.43	-17.25	
1.6	17.03	0	17.03	15.55	-17.15	
1.7	17.02	0	17.02	17.66	-17.10	
1.8	17.01	0	17.01	19.76	-17.06	
1.9	17.01	0	17.01	21.76	-17.04	
2.0	17.01	0	17.01	23.85	-17.03	

Column (1) gives values of v in the range of interest. Column (2) is H_o; in the range of v from zero to one it drops off at a constant slope of thirty decibels per decade, which is not an integral multiple of twenty decibels per decade. Column (3) gives the first component selected for inclusion in $-A_o$. It is a standard component of type II from Appendix B, having for the coefficient 2c a value of -0.4151, with origin at v equals zero, negative sign, and dropped 0.50 decibels below the level of the normalized standard component given in the tables. This particular standard component was selected to make $-A_o$ remain as close to H_o as possible at the critical break point at v equals zero, and still fall within the tolerance bound of -0.50 at low frequencies. Note that the standard component from Appendix B having the next smaller (more negative) value of 2c, -0.7411, has a maximum deviation of 0.64 at v equals -0.2. If this component were chosen it would have to be dropped at least 0.64 db. in order to remain below H_o, and then it would fall below the -0.50 tolerance limit at low frequencies.

The next component of $-A_o$ is given in column (4). It is a type I function from Appendix B, with positive sign, and origin displaced to v equals 0.5. Column (5) gives $-A_o$, and is the sum of columns (3) and (4). Then

$$-A_o = -0.5 - 10\log(1 - 0.4151\,\omega^2 + \omega^4) + 10\log\left(1 + \frac{\omega^2}{(10^{0.5})^2}\right)$$

and

$$a_o = \frac{11.220\,\omega^4 - 4.6576\,\omega^2 + 11.220}{\omega^2 + 10}$$

Column (6) is $A_o + H_o$, a useful column of values to record for later reference in determining when a successful approximant has been achieved. Column (7) gives the values of H_1, the first remainder. As indicated by the symbol at the top of the column, the values of H_1 are all complex. The intermediate values necessarily obtained in proceeding from column (6) to column (7), which were recorded in the first example, are omitted here for brevity. H_1 is plotted graphically in Figure 43.

Before the choice of $-A_1$ what has been accomplished so far will be reviewed. From column (6) note that for all v equal to or less than -0.2 and for all v equal to or greater than 1.0, $-A_o$ falls within the -0.50 decibel tolerance limit from H_o. According to inequality (23) this means that as long as succeeding $-A_k$ are chosen to fall below the corresponding H_k, all succeeding approximants will be satisfactory in these ranges. Therefore in choosing $-A_1$ we should concentrate on obtaining a good approximation in the range from -0.1 to 0.9, inclusive, and do not need to worry about the form of $-A_1$ outside this range except to keep it below H_1.

In selecting components for $-A_1$ to approximate H_1, a compromise between fitting the peak at 0.9 and the sloping portion in the vicinity of 0.3 suggests the type II standard component with 2c = -0.7411, origin displaced to 0.6, high and low frequency ends exchanged, negative sign, and raised 17 decibels. This function is transcribed from the tables of Appendix B, as modified, to column (8). In order to come closer to H_1 at the low frequency end a type I function, with origin at -0.2, positive sign, and high and low frequency ends exchanged, is recorded in column (9). Columns (8) and (9) are added to give $-A_1$ in column (10). Note that because H_1 carries the I symbol $-A_1$ must also be accompanied

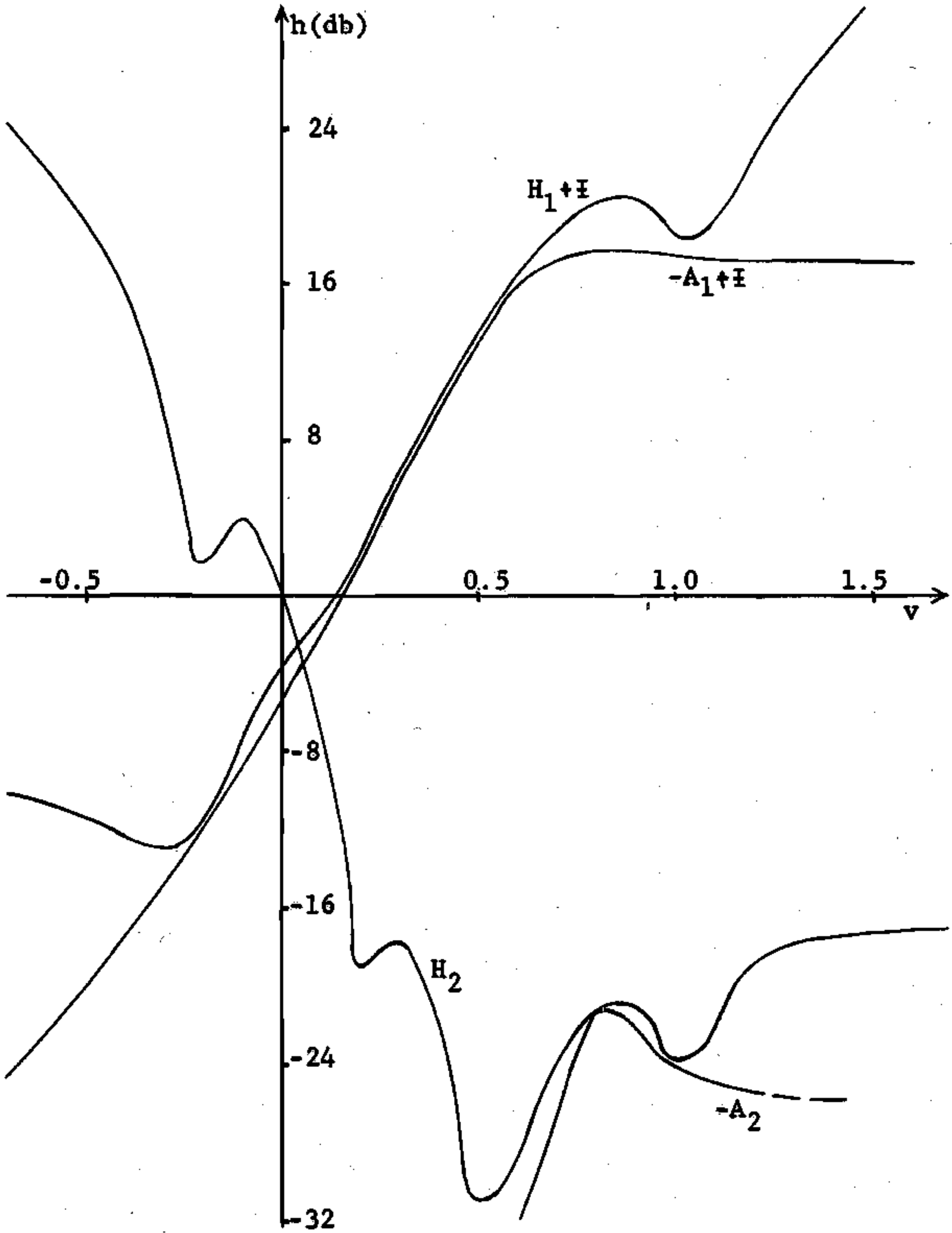

Fig. 43. Second Example Continued.

by it, as indicated at the top of column (10). From its composition indicated above,

$$-A_1 = I + 17.00 - 10\log\left(\frac{(10^{0.6})^4}{\omega^4} - \frac{0.7411(10^{0.6})^2}{\omega^2} + 1\right)$$
$$+ 10\log\left(\frac{(10^{-0.2})^2}{\omega^2} + 1\right)$$

and

$$a_1 = -\frac{\omega^4 - 11.745\,\omega^2 + 251.19}{50.119\,\omega^4 + 19.953\,\omega^2}$$

$A_1 + H_1$ is tabulated in column (11) and H_2 is computed and recorded in column (12), intermediate steps in the computation again being omitted.

In practice it would not be necessary to compute the full range of values of H_2. Table 1 of Chapter V indicates that when H_k is very much greater than $-A_k$, H_{k+1} is approximately equal to $A_k + I$. This approximate relationship may be used to sketch the general position of H_2 at the ends of the frequency range of interest; then if the next element is well below this general position, the requirement that $A_2 + H_2$ be positive ($a_2G_2 > 1$) is met without the need for computing points of H_2 in these non-critical regions. In this example the full range of H_2 has been computed merely to verify the prediction of Table 1.

Before proceeding we again review the situation. The range of interest is from -0.1 to 0.9. In that range $A_o + H_o$ (column (6)) is less than 3.00, and from 0.2 to 0.6 inclusive $A_1 + H_1$ (column (11)) is less than 0.50. From inequality (26) of Chapter III it follows that H_{o1} is satisfactory in the range from 0.2 to 0.6. We already know it is satisfactory for logarithmic frequencies less than -0.1 and greater than 0.9. That leaves six points to be checked, those corresponding to values of v of -0.1, 0, 0.1, 0.7, 0.8, and 0.9. As might be expected these are in the general vicinity of the sharp breaks in the prescribed function. Appendix D may be consulted in an attempt to eliminate some of these points from consideration. From Figure 58 of Appendix D, for D = -0.50 and n = 1, the critical value of M is about 1.45. This value is compared with columns (6) and (11) at the six points in question; for five of them the $A_k + H_k$ values fall on opposite sides of 1.45, giving indeterminate results; but for v = 0.8, $A_o + H_o = 1.49$ and $A_1 + H_1 = 2.44$, both larger than 1.45. This indicates that H_{o1} is <u>not</u> satisfactory at 0.8. At the other points H_{o1} must be evaluated using equation (14) of Chapter II for n = 1.

$$H_{o1} = -A_o - L(-A_o + I - A_1)$$

Upon performing the calculations the following results are obtained. At v = -0.1,

$$H_{o1} = -0.79 - L(-0.79 + I - (8.76 + I))$$

$$= -0.79 - L(-9.55)$$

$$= -0.79 - (-0.51) = -0.28$$

Since $H_o = 0$ at this point, H_{o1} falls within the prescribed tolerance of -0.50 db. and is a satisfactory approximant. At the other logarithmic frequencies in question the following are obtained.

v	H_{o1}	H_o	Approximant
0	- 1.23	- 0.75	satisfactory
0.1	- 3.22	- 3.00	satisfactory
0.7	-21.58	-21.00	0.08 db below the limit
0.8	-24.70	-24.00	0.20 db below the limit (confirming the result from Appendix D)
0.9	-27.39	-27.00	satisfactory

In order to produce a satisfactory approximant one more element must be chosen. The shape of H_2 in the vicinity of 0.8 suggests a standard component of type II. Such a component is selected with $2c = -1.7488$, origin at 0.8, negative sign, high and low frequency ends reversed, and lowered by a constant 27.35 db; the critical values involved are given in column (13) and that portion of $-A_2$ is plotted on Figure 43. Checking out the point at 0.7,

$$H_{o2} = - 22.98 - L(-22.98 + \mathrm{I} + 17.38 + \mathrm{I} - L(+ 17.38$$

$$+ \mathrm{I} + \mathrm{I} - 26.04))$$

$$= - 21.31 \quad \text{and is therefore satisfactory.}$$

If a type I function were tried for $-A_2$ (in an effort to keep down the order of the final approximant) it would be found that the approximation to H_2 is not good enough at 0.8 to produce a satisfactory approximant. From its composition indicated above,

$$-A_2 = -27.35 - 10\log\left(\frac{(10^{0.8})^4}{\omega^4} - \frac{1.7488(10^{0.8})^2}{\omega^2} + 1\right)$$

and

$$a_2 = \frac{543.25\,\omega^4 - 37876\,\omega^2 + 860994}{\omega^4}$$

We may now construct G_{o2} from the continued fraction form,

$$G_{o2} = \cfrac{1}{a_o + \cfrac{1}{a_1 + \cfrac{1}{a_2}}}$$

$$G_{o2} = \frac{0.088804\,\omega^{10} - 7.0854\,\omega^8 + 180.02\,\omega^6 - 938.67\,\omega^4 + 3584.6\,\omega^2 + 389470}{\omega^{12} - 94.782\,\omega^{10} + 3243.5\,\omega^8 - 45212\,\omega^6 + 376930\,\omega^4 - 252010\,\omega^2 + 436990}$$

<u>Third example: approximation with general rational functions</u>.--In this example, as in the previous one, $-A_k$ are chosen less than H_k. The example illustrates the method of carrying forward both tolerance bounds from stage to stage to determine when a satisfactory approximant has been reached. This permits the non-uniform tolerances of the prescribed function to be considered in forming the approximant. The upper and lower bounds of the prescribed function are portrayed in Figure 44. Pertinent values are given in Table 5 below, intermediate steps in the calculation being omitted.

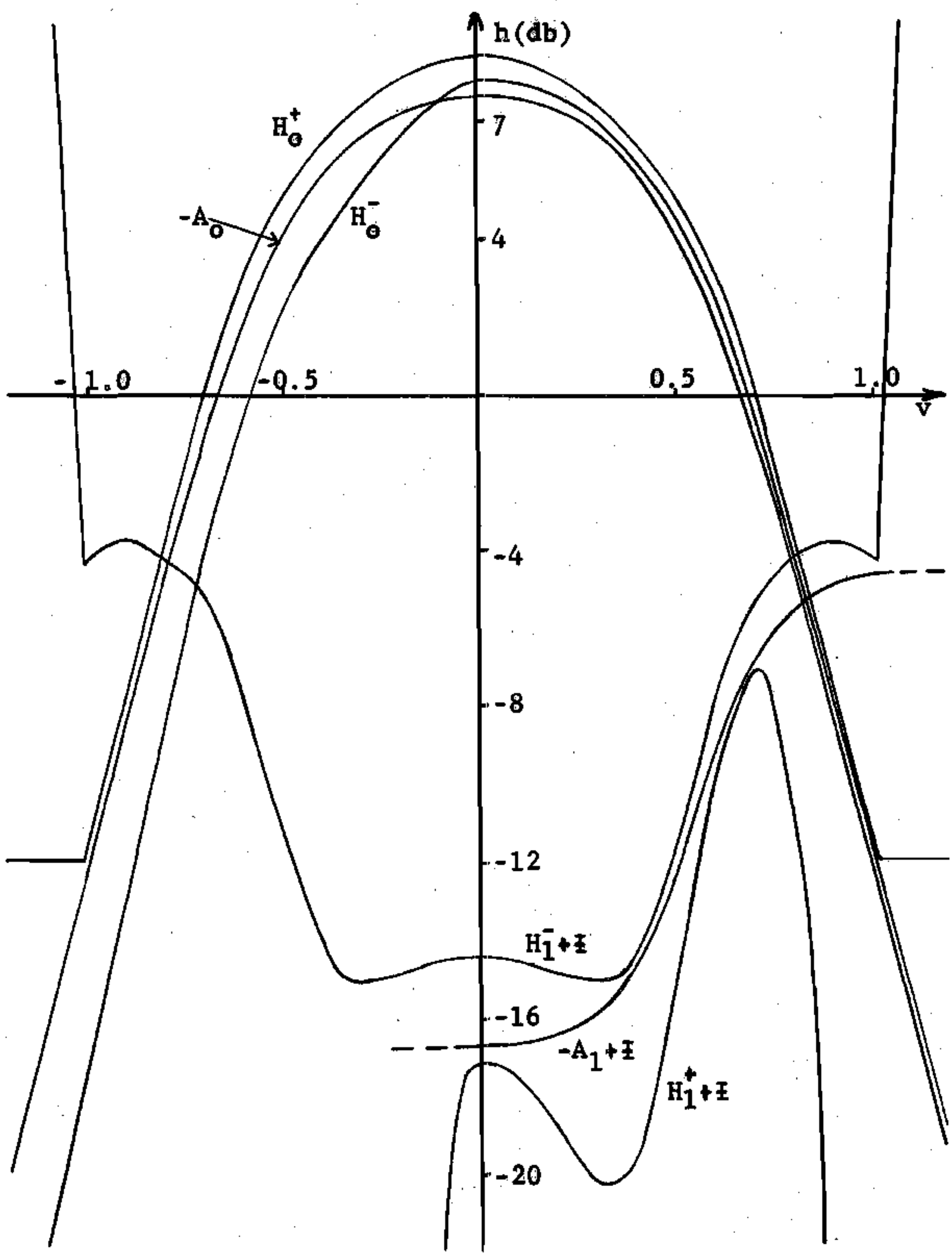

Fig. 44. Third Example.

Table 5. Values of Quantities Pertaining to Third Example.

v	H_o^+	H_o^-	$-A_o$	H_1^- / I+	H_1^+ / I+	$-A_1$ / I+
-1.2	-12.00	-26.50	-19.94	19.18		
-1.1	-12.00	-22.00	-16.00	13.80		
-1.0	-12.00	-17.50	-12.10	- 4.33		
-0.9	- 8.00	-13.00	- 8.28	- 3.77		
-0.8	- 4.00	- 8.50	- 4.61	- 4.22		
-0.7	0	- 4.00	- 1.19	- 5.01		
-0.6	3.05	- 0.45	1.79	- 7.78		
-0.5	5.04	2.04	4.14	-11.42		
-0.4	6.45	3.95	5.76	-14.09		
-0.3	7.46	5.46	6.76	-15.03		
-0.2	8.13	6.63	7.30	-14.90		
-0.1	8.52	7.52	7.56	-14.59		
0	8.65	8.15	7.64	-14.47	-17.19	-16.81
0.1	8.52	8.02	7.56	-14.59	-17.54	-16.78
0.2	8.13	7.63	7.30	-14.90	-18.66	-16.59
0.3	7.46	6.96	6.76	-15.03	-20.23	-15.97
0.4	6.45	5.95	5.76	-14.09	-19.44	-14.52
0.5	5.04	4.54	4.14	-11.42	-14.70	-12.11
0.6	3.05	2.55	1.79	- 7.78	- 9.73	- 9.26
0.7	0	- 0.50	- 1.19	- 5.01	- 7.14	- 6.88
0.8	- 4.00	- 4.50	- 4.61	- 4.22	-11.41	- 5.44
0.9	- 8.00	- 8.50	- 8.28	- 3.77		- 4.88
1.0	-12.00	-12.50	-12.10	- 4.33		- 4.62
1.1	-12.00	-16.50	-16.00	13.80		
1.2	-12.00	-20.50	-19.94	19.18		

$-A_o$ has the following components.

$$-A_o = 8.16 - 10\log\left(1 + \frac{0.5119\,\omega^2}{(10^{0.5})^2} + \frac{\omega^4}{10^{0.5})^4}\right)$$

$$- 10\log\left(\frac{(10^{-0.5})^4}{\omega^4} + \frac{0.5119(10^{-0.5})^2}{\omega^2} + 1\right)$$

If type II functions with 2c equal to zero are used instead of the above it is found that the dips in $H_1^- + I$ at v = -0.4 and v = 0.4 are accentuated to the extent that the order of $-A_1$ must be greater than the one obtained in this example, if $-A_1$ is to fall within the bounds on H_1. In selecting the constant 8.16 in $-A_o$ above, use was made of equation (43) to give $H_1^- + I$ approximately a zero slope in the region of 0.8, 0.9, and 1.0. If $-A_o$ is brought closer to H_o^+ a sharp dip develops in $H_1^- + I$ at $v = 1.0$, which would also require a higher order in $-A_1$.

$-A_o$ is symmetrical about $v = 0$, as is H_o^+, but in the selection of $-A_1$ Figure 42 clearly shows the use made of the unsymmetrical lower tolerance bound. Note also that the computations are shortened by omitting those portions of H_1^+ which fall on the real logarithmic sheet. $-A_1$ is composed as follows:

$$-A_1 = I - 16.70 + 10 \log\left(1 - \frac{0.4151\,\omega^2}{(10^{0.4})^2} + \frac{\omega^4}{(10^{0.4})^4}\right)$$

$$- 10 \log\left(1 - \frac{0.4151\,\omega^2}{(10^{0.7})^2} + \frac{\omega^4}{(10^{0.7})^4}\right.$$

From $-A_o$ and $-A_1$ above,

$$a_o = \frac{1.52757\,\omega^8 + 7.89761\,\omega^6 + 153.173\,\omega^4 + 7.89761\,\omega^2 + 1.52757}{1000\,\omega^4}$$

$$a_1 = -\frac{0.0741310\,\omega^4 - 0.772964\,\omega^2 + 46.7735}{0.0251189\,\omega^4 - 0.0657901\,\omega^2 + 1}$$

$$G_{o1} = \frac{654.64\,\omega^8 - 6825.9\,\omega^6 + 413050\,\omega^4}{\omega^{12} - 5.2568\,\omega^{10} + 455.51\,\omega^8 + 2802.7\,\omega^6 + 54384\,\omega^4 + 3251.7\,\omega^2 + 630.96}$$

H_{o1} is plotted in Figure 45, together with the tolerance bounds of the prescribed function. In Figure 46, H_o^+ is portrayed as a horizontal line, and H_{o1} and H_o^- are plotted at the appropriate values below H_o^+. This manner of portraying the above quantities permits the use of an expanded ordinate scale, and emphasizes how H_{o1} fluctuates in the acceptable band between the two tolerance bounds.

Fourth example: approximation with linear functions asymptotic at infinity.--A function is prescribed over a range of 3.6 decades with a $\pm$ 0.50 decibel tolerance; its upper and lower bounds are plotted in Figure 47 and their amplitudes are listed in the tabulation below. The function is assumed to be linear at twenty decibels per decade at lower frequencies and at minus forty decibels per decade at higher frequencies.

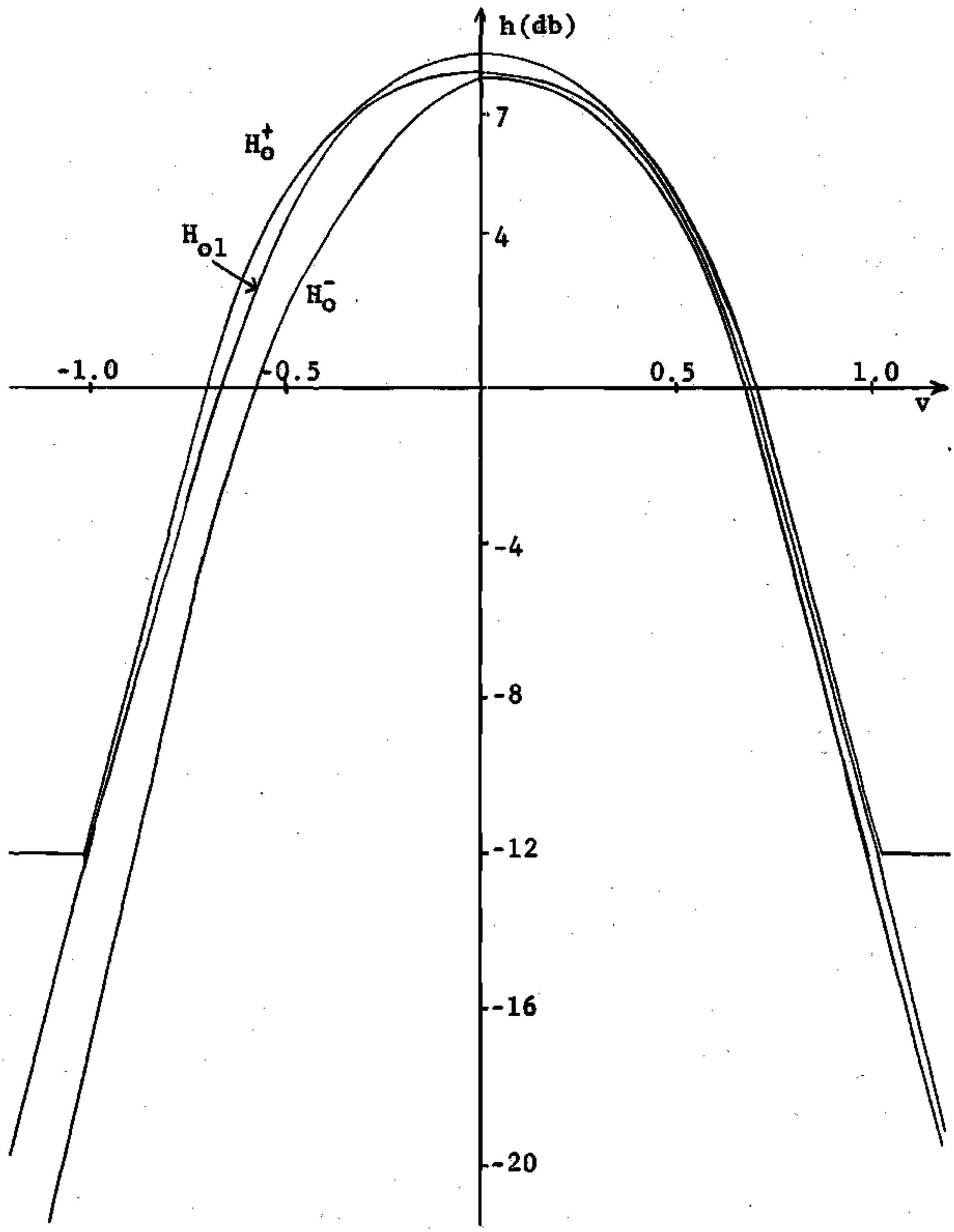

Fig. 45. Third Example Continued.

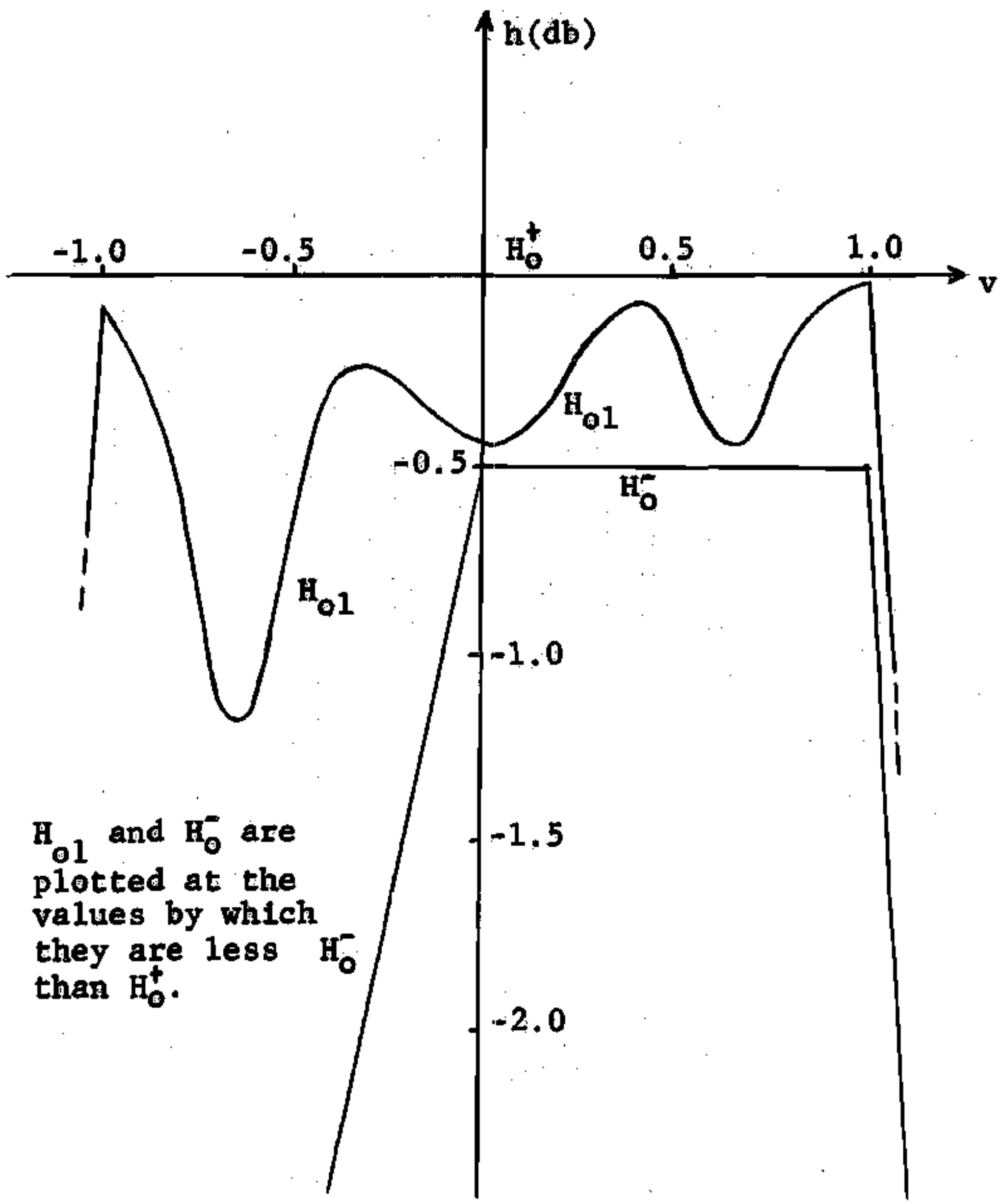

Fig. 46. Third Example Continued.

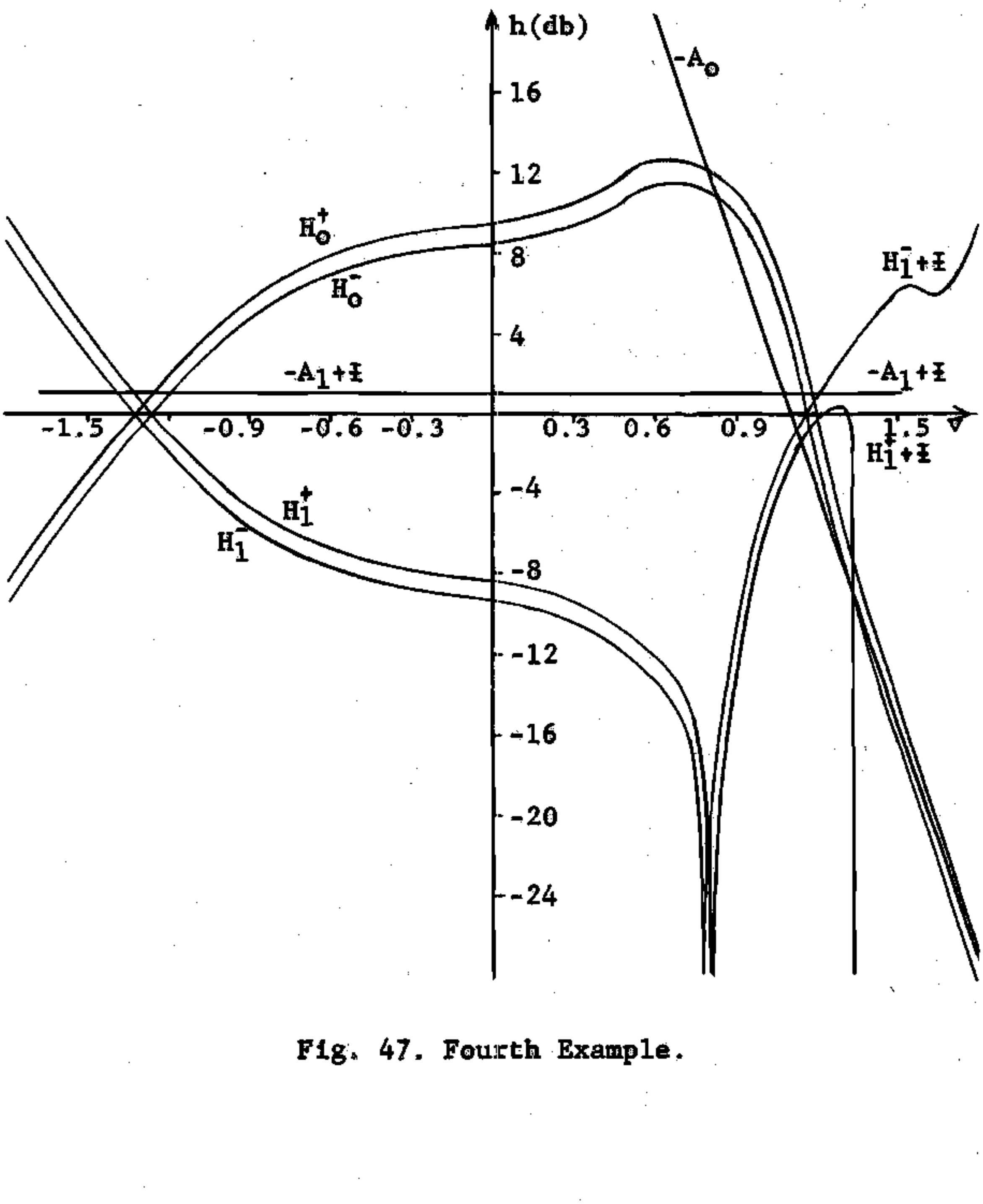

Fig. 47. Fourth Example.

Table 6. Values of Quantities Pertaining to Fourth Example

v	H_o^+	H_o^-	H_1^-	H_1^+	H_2^+	H_2^-	H_3^-	H_3^+
-1.8	- 8.88	- 9.88	8.88	9.88	- 0.34	- 0.47	0.34	0.47
-1.7	- 6.88	- 7.88	6.88	7.88	0	- 0.19	0	0.19
-1.6	- 4.88	- 5.88	4.88	5.88	0.49	0.28	- 0.49	- 0.28
-1.5	- 3.00	- 4.00	3.00	4.00	1.12	0.76	- 1.12	- 0.76
-1.4	- 1.32	- 2.32	1.32	2.32	1.85	1.40	- 1.85	- 1.40
-1.3	0.32	- 0.68	- 0.32	0.68	2.72	2.17	- 2.72	- 2.17
-1.2	1.84	0.84	- 1.84	- 0.84	3.66	3.03	- 3.66	- 3.03
-1.1	3.40	2.40	- 3.40	- 2.40	4.75	4.03	- 4.75	- 4.03
-1.0	4.76	3.76	- 4.76	- 3.76	5.78	5.01	- 5.78	- 5.01
-0.9	5.80	4.80	- 5.80	- 4.80	6.62	5.81	- 6.62	- 5.81
-0.8	6.80	5.80	- 6.80	- 5.80	7.47	6.62	- 7.47	- 6.62
-0.7	7.48	6.48	- 7.48	- 6.48	8.06	7.19	- 8.06	- 7.19
-0.6	8.00	7.00	- 8.00	- 7.00	8.51	7.64	- 8.51	- 7.64
-0.5	8.44	7.44	- 8.44	- 7.44	8.91	8.02	- 8.91	- 8.02
-0.4	8.76	7.76	- 8.76	- 7.76	9.20	8.30	- 9.20	- 8.30
-0.3	9.00	8.00	- 9.00	- 8.00	9.41	8.51	- 9.41	- 8.51
-0.2	9.16	8.16	- 9.16	- 8.16	9.56	8.66	- 9.56	- 8.66
-0.1	9.24	8.24	- 9.24	- 8.24	9.63	8.73	- 9.63	- 8.73
0	9.36	8.36	- 9.36	- 8.36	9.74	8.84	- 9.74	- 8.84
0.1	9.56	8.56	- 9.56	- 8.56	9.93	9.02	- 9.93	- 9.02
0.2	9.80	8.80	- 9.81	- 8.81	10.16	9.24	-10.16	- 9.24
0.3	10.28	9.28	-10.31	- 9.30	10.62	9.69	-10.62	- 9.69
0.4	11.00	10.00	-11.09	-10.07	11.35	10.40	-11.34	-10.40
0.5	11.76	10.76	-12.03	-10.97	12.24	11.24	-12.21	-11.22
0.6	12.52	11.52	-13.39	-12.20	13.55	12.40	-13.41	-12.29
0.7	12.76	11.76	-15.62	-13.86	15.71	14.00	-14.87	-13.42
0.8	12.40	11.40	-21.82	-20.80	21.80	20.83	-17.92	-15.08
0.9	11.16	10.16	-10.73	-11.89	10.43	11.66	- ∞	-16.51
1.0	9.20	8.20	- 5.46	- 5.97	4.35	5.00		-13.53
1.1	5.10	4.10	- 1.50	- 2.03	- 2.09	- 0.96		- 7.26
1.2	- 1.00	- 2.00	1.12	- 0.14	- ∞	- 6.23		1.81
1.3	- 6.30	- 7.30	3.32	0.18		- 7.82		12.23
1.4	-11.10	-12.10	5.08	- ∞		+ ∞		19.57
1.5	-15.60	-16.60	6.18					25.57
1.6	-19.90	-20.90	5.91					31.57
1.7	-24.00	-25.00	6.11					37.57
1.8	-28.00	-29.00	10.11					43.57

The values in these four columns below the horizontal lines carry the ∓ symbol

Table 6. Values of Quantities Pertaining to Fourth Example (continued)

v	H_4^+	H_4^-	H_5^-	H_5^+	H_6^+	H_6^-
-1.8	- 0.34	-0.47	- 0.24	- 0.09	- 0.43	- 0.61
-1.7	0	-0.19	- 0.63	- 0.44	- 0.38	- 0.62
-1.6	0.49	0.28	- 1.18	- 0.95	- 0.31	- 0.63
-1.5	1.12	0.76	- 1.95	- 1.52	- 0.16	- 0.88
-1.4	1.85	1.40	- 2.85	- 2.29	- 0.12	- 1.31
-1.3	2.71	2.16	- 3.97	- 3.25	- 0.08	- 2.22
-1.2	3.65	3.02	- 5.28	- 4.39	- 0.11	- 4.57
-1.1	4.74	4.02	- 6.97	- 5.83	0.70	-15.79
-1.0	5.77	5.00	- 8.86	- 7.42	2.22	- ∞
-0.9	6.61	5.79	-10.79	- 8.90	3.47	
-0.8	7.45	6.60	-13.47	-10.76	8.51	
-0.7	8.04	7.16	-16.55	-12.41	13.26	
-0.6	8.48	7.60	-21.54	-14.10	20.19	
-0.5	8.86	7.96	- ∞	-16.01	+ ∞	
-0.4	9.13	8.21		-17.93		
-0.3	9.30	8.37		-19.73		
-0.2	9.39	8.45		-20.97		
-0.1	9.36	8.40		-20.16		
0	9.32	8.31		-18.97		
0.1	9.27	8.19		-17.74		
0.2	9.13	7.92		-15.76		
0.3	9.06	7.65		-14.33		
0.4	9.08	7.36		-13.12		
0.5	8.97	6.53		-10.58		
0.6	9.06	4.88		- 8.09		
0.7	9.64	- ∞		+ ∞		
0.8	14.33					
0.9	+ ∞					

The first element, $-A_o$, is chosen to be 43.93 - 40v db. This places it within 0.07 db. of the upper bound, H_o^+, at high frequencies. To bring it much closer would accentuate the dip at the high-frequency end of H_1^- enough to block the selection of a constant for $-A_1$. The bounds of H_1 are also shown on Figure 47, and the values are given, along with those of succeeding remainders, in the tabulation above. The zeros of the H_1 bounds (negative infinite values) are due to the fact that $-A_o$ crosses the H_o bounds. Note also that a segment of the high-frequency end of H_1^+ has been replaced by minus infinity, which is permissible since the values replaced are on the opposite logarithmic sheet and are not of interest. However, no omissions may be made in the vicinity of the singularities near the abscissa 0.8.

The element $-A_1$ is chosen equal to $1.00 + \text{I}$. It must be complex because H_1^+ and H_1^- are complex at at high end of the frequency range. H_2 and H_3 are shown on Figure 48. For $-A_2$ the function $64.43 + \text{I} - 60\ v$ is chosen. In calculating H_3 care must be taken with signs and complex symbols in the vicinity of 0.8. Note that the poles in H_2 disappear in H_3. This follows from the formula,

$$H_3 = \text{I} + A_2 + L(-H_2 - A_2),$$

since $L(-\infty)$ equals zero. $-A_3$ is $0.58 - 20v$.

Figure 49 illustrates the pertinent segments of the bounds of H_4, H_5, and H_6. $-A_4$ is 8.70; $-A_5$ is $-27.80 - 20v$; and $-A_6$ is -0.508. As $-A_6$ falls within the tolerance bounds of H_6 throughout the entire frequency range, the expansion is terminated. The following elements have been obtained.

$-A_o = 43.93 - 40\ v$	$a_o = 40.457(10^{-6})\,\omega^4$
$-A_1 = 1.00 + \text{I}$	$a_1 = -\ 0.79433$
$-A_2 = 64.43 + \text{I} - 60v$	$a_2 = -\ 3.6058(10^{-7})\,\omega^6$
$-A_3 = 0.58 - 20v$	$a_3 = 0.87498\,\omega^2$
$-A_4 = 8.70$	$a_4 = 0.13490$
$-A_5 = -\ 27.80 - 20v$	$a_5 = 602.56\,\omega^2$
$-A_6 = -0.508$	$a_6 = 1.1241$

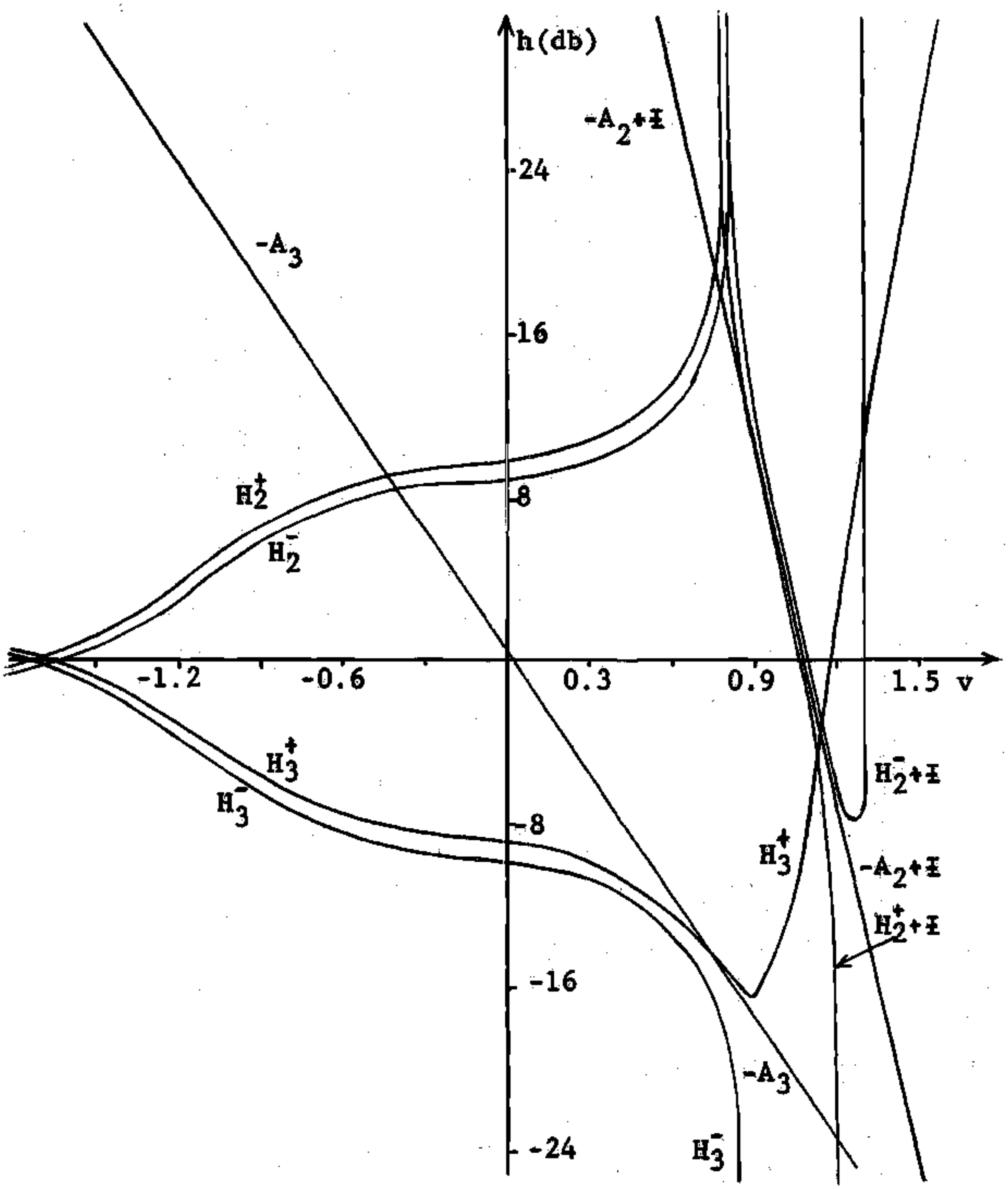

Fig. 48. Fourth Example Continued.

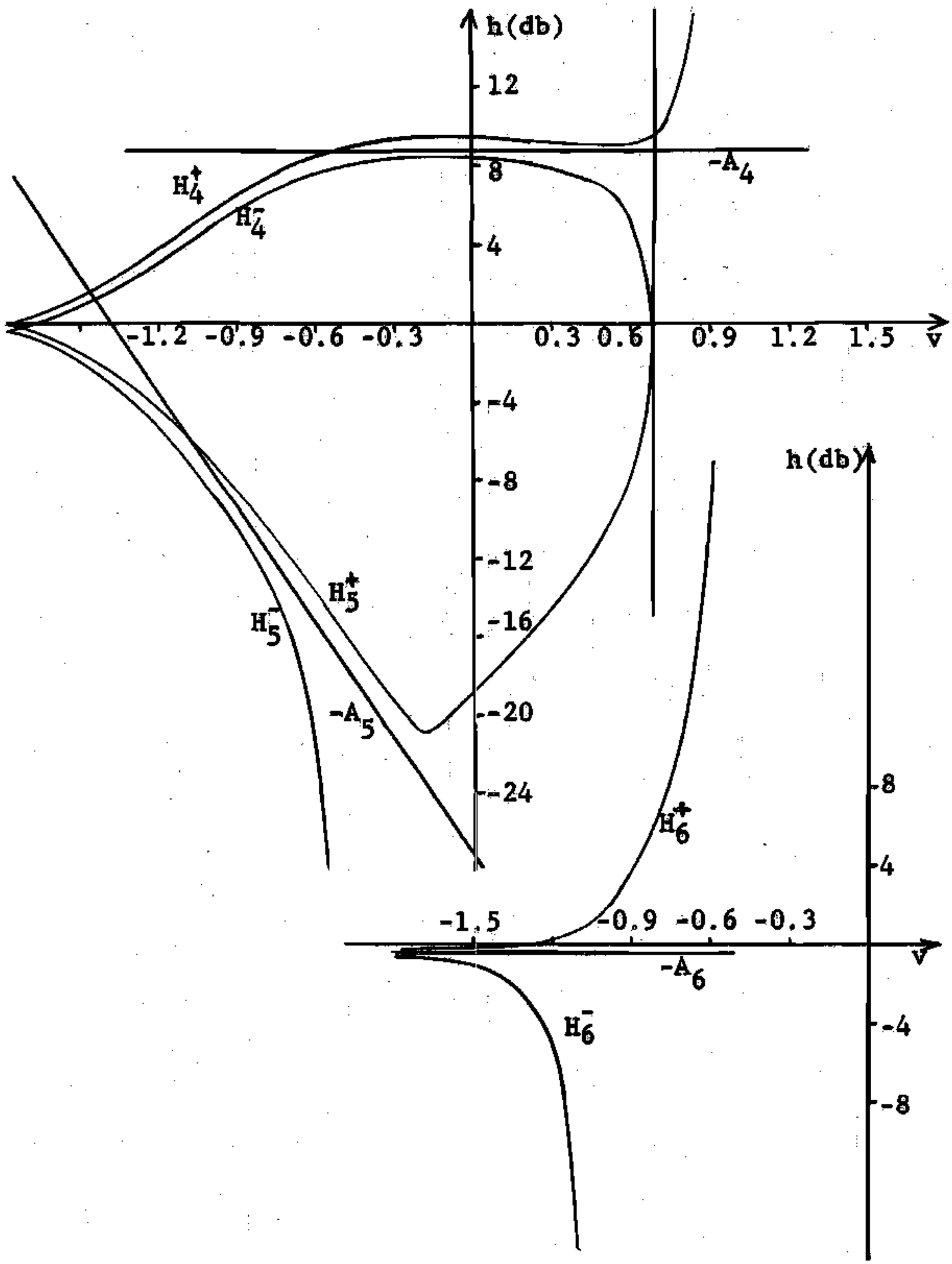

Fig. 49. Fourth Example Continued.

The selection of $-A_6$ is of particular importance. Note that H_o approaches minus infinity at twenty decibels per decade for very small frequencies and H_1 approaches plus infinity in the same range. The selection of $-A_1$ (1.00 + I) placed it far below H_1 for very low frequencies. From Table 1 this means that H_2^+ and H_2^- both approach -1.00 at low frequencies. The same table also indicates that the low-frequency asymptote of both H_3^+ and H_3^- is 1.00, and that of the H_4 bounds is -1.00. $-A_4$ is a constant so the low-frequency asymptote of H_5 is calculated as follows.

$$H_5 = - H_4 + L\,(H_4 + A_4)$$

$$= 1.00 + L(-9.70) = 0.508$$

The asymptote of H_6 is -0.508 since $-A_5 \gg H_5$ at low frequencies. $-A_6$ must then be chosen equal to -0.508. Let us now examine the result of this selection by forming the continued fraction with zero substituted for ω.

$$G_{o6} = \cfrac{1}{0 + \cfrac{1}{-0.79433 + \cfrac{1}{0 + \cfrac{1}{0 + \cfrac{1}{0.13490 + \cfrac{1}{0 + \cfrac{1}{1.1241}}}}}}}$$

$$= - 0.79433 + \frac{1}{0.1349 + 1.1241}$$

$= - 0.79433 + 0.79428 \approx 0$, as it should be.

The desired approximant, expressed in the form of a rational fraction, is:

$$G_{o6} = \frac{2.4718(10^4)\,\omega^{10} + 2.0974(10^5)\,\omega^8 + 309.03\,\omega^6 + 8.6299(10^{10})\,\omega^4 + 6.5397(10^{11})\,\omega^2}{\omega^{14} + 8.4856\,\omega^{12} - 3.1117(10^4)\,\omega^{10} + 3.2273(10^6)\,\omega^8 + 2.6458(10^7)\,\omega^6 + 9.8630(10^{10})\,\omega^2 + 1.3590(10^9)}$$

H_{o6} is plotted in Figure 50 for comparison with the prescribed tolerance bounds.

Fifth example: approximation with linear elements asymptotic at an intermediate frequency.--The tolerance bounds of a prescribed function are given in Figure 51 and it is desired to approximate it with particular attention to the behavior in the vicinity of ω equals one ($v = 0$). Letting

$$\lambda^2 = \omega^2 - 1$$

$$\log \lambda = V$$

and making use of equation (52) and the relation

$$V + \frac{1}{20} = V',$$

the function is tabulated against the variable V or V' and replotted in Figure 52. The values of the prescribed tolerance bounds and those of successive remainders are given in the tabulation below.

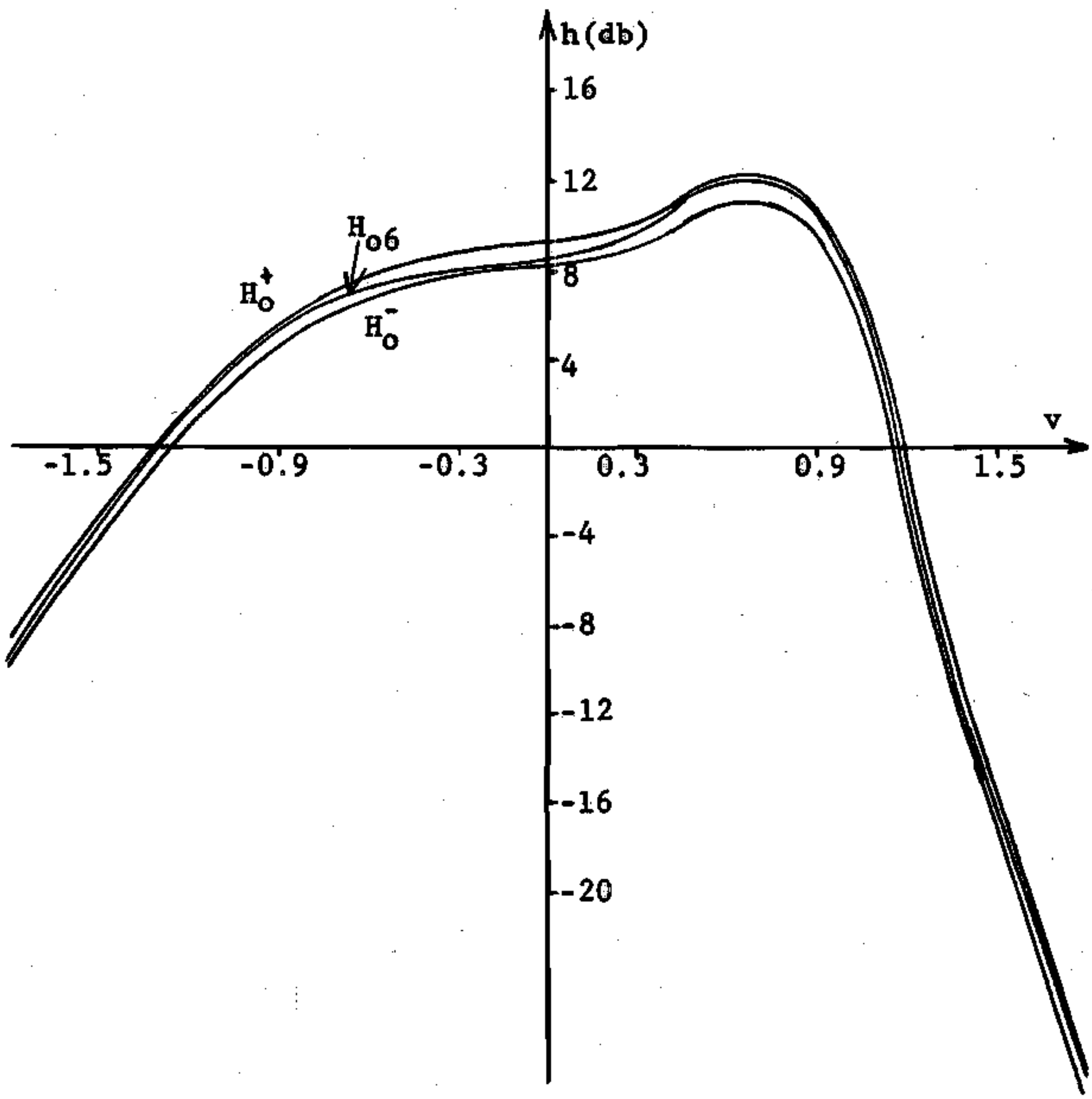

Fig. 50. Fourth Example Continued.

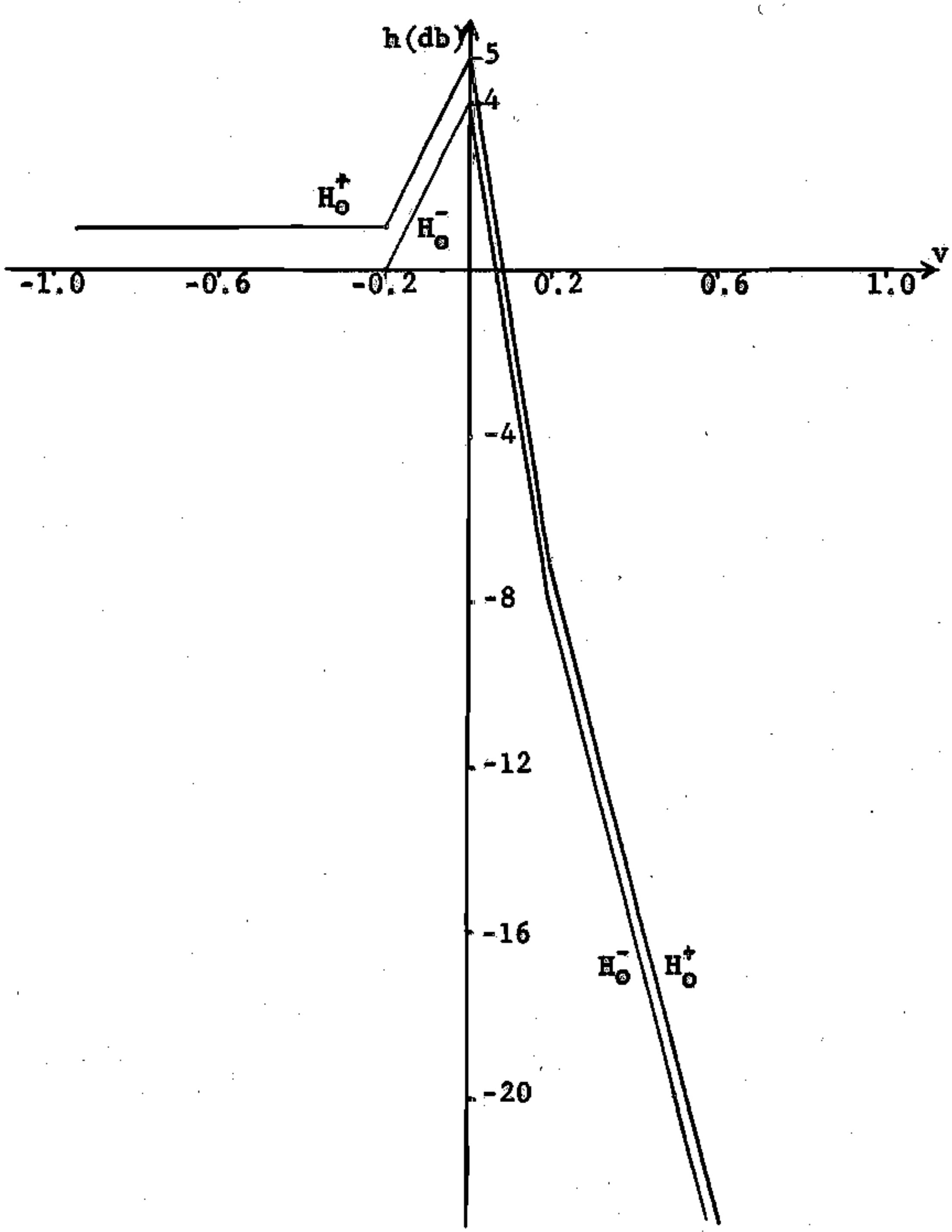

Fig. 51. Fifth Example.

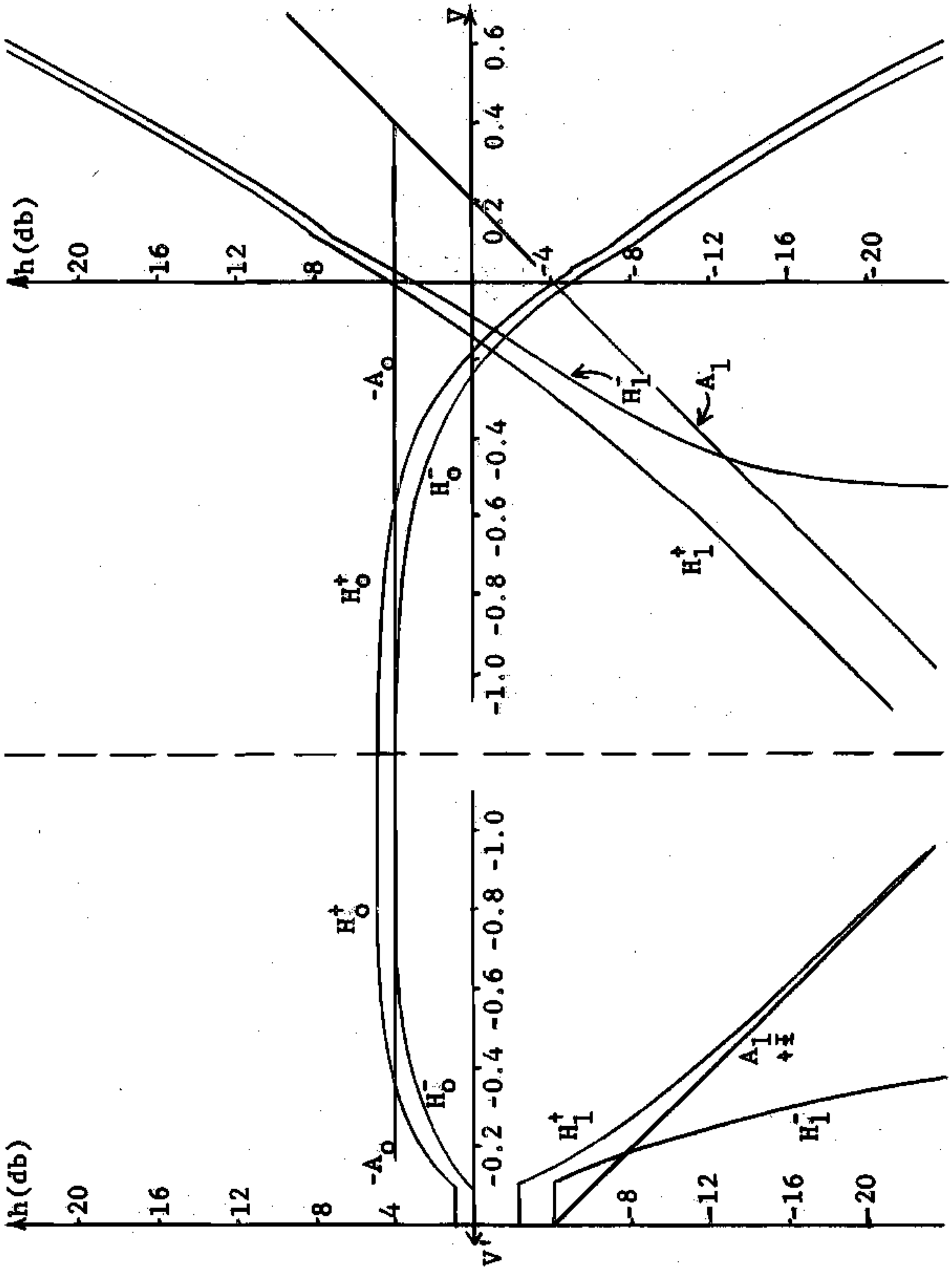

Fig. 52. Fifth Example Continued.

Table 7. Values of Quantities Pertaining to Fifth Example

v	V'	H_0^+	H_0^-	H_1^-	H_1^+	H_2^-	H_2^+
$-\infty$	0	1.00	0	-4.02	-2.20	-1.00	0
-0.216	-0.1	1.00	0	-4.02	-2.20	-1.89	-0.69
-0.110	-0.2	2.80	1.80	-8.97	-5.81	-5.45	-3.76
-0.063	-0.3	3.74	2.74	-16.10	-8.73	-9.05	-6.31
-0.037	-0.4	4.26	3.26	-16.36I*	-11.31	-13.98*	-8.63
-0.023	-0.5	4.53	3.53	-13.40I*	-13.42	$-\infty$ *	-10.69
-0.014	-0.6	4.72	3.72		-15.77		-12.87
-0.009	-0.7	4.82	3.82		-17.73		-14.85
-0.0055	-0.8	4.89	3.89		-19.91		-16.94
-0.0035	-0.9	4.93	3.93		-21.89		-18.93
-0.0022	-1.0	4.96	3.96		-24.34		-21.16
0	$-\infty$	5.00	4.00				
	V						
0	$-\infty$	5.00	4.00				
0.0022	-1.0	4.87	3.87		-19.17		-20.90
0.0034	-0.9	4.80	3.80		-17.27		-19.05
0.0054	-0.8	4.68	3.68		-15.17		-16.90
0.008	-0.7	4.52	3.52		-13.32		-15.13
0.013	-0.6	4.22	3.22	$-\infty$	-11.06		-12.74
0.020	-0.5	3.80	2.80	-17.27	-8.97	$-\infty$	-10.61
0.032	-0.4	3.08	2.08	-10.27	-6.55	-15.10	-8.01
0.049	-0.3	2.06	1.06	-6.49	-4.14	-9.05	-5.44
0.073	-0.2	0.62	-0.38	-3.29	-1.59	-5.08	-2.72
0.106	-0.1	-1.36	-2.36	-0.13	1.22	-1.43	0.29
0.150	0	-4.00	-5.00	3.25	4.42	2.34	3.75
0.206	0.1	-7.24	-8.24	6.90	7.97	6.30	7.51
0.273	0.2	-9.92	-10.92	9.74	10.78	9.25	10.40
0.348	0.3	-12.92	-13.92	12.83	13.86	12.46	13.57
0.432	0.4	-16.28	-17.28	16.24	17.25	15.97	17.04
0.521	0.5	-19.84	-20.84	19.82	20.83	19.64	20.68
0.613	0.6	-23.52	-24.52	23.51	24.51	23.39	24.41
0.709	0.7	-27.34	-28.34	27.32	28.32	27.24	28.26

Table 7. Values of Quantities Pertaining to Fifth Example (continued)

V'	H_3^+	H_3^-	H_4^-	H_4^+	H_5^-	H_5^+
0	- 0.16	- 1.52	1.69I	0.39I	0.15	1.52
-0.1	- 0.95	- 3.66	1.03I	- 7.23I	- 4.99	1.45
-0.2	2.16	- 2.88	2.80I	- 2.43I	- ∞	- 2.68
-0.3	5.26	- ∞	3.45I	- ∞		- 2.00
-0.4	11.22		3.87I			- 4.00
-0.5	+ ∞		4.00I			- 6.00
- ∞			4.00I			
V						
-0.5	+ ∞					
-0.4	13.14					
-0.3	5.26	- ∞		+ ∞		+ ∞
-0.2	1.33	-20.66		20.57		20.53
-0.1	- 1.88	- 7.12	- ∞	4.22		0.24
0	- 5.40	- 8.98	0.20	7.32	- ∞	4.60
0.1	- 9.32	-12.22	7.81	11.51	3.03	10.08
0.2	-11.42	-13.52	10.55	13.01	7.02	11.36
0.3	-14.19	-15.96	13.75	15.67	11.37	14.30
0.4	-17.48	-19.08	17.28	18.94	15.74	17.96
0.5	-21.02	-22.52	20.93	22.48	19.95	21.77
0.6	-24.68	-26.11	24.64	26.08	24.00	25.68
0.7	-28.48	-29.89	28.46	29.88	28.05	29.59

Consideration of Figure 52 indicates that a good choice for $-A_o$ is 4.00. Also from the same figure the probable slope, at ω equals unity, of a satisfactory approximant is minus twenty decibels per decade, whereas the slope of H_o^- for small negative v is plus twenty decibels per decade. Since H_1^+ is the first remainder derived from H_o^-, we choose $-A_1$ to be asymptotic to H_1^+ as V' approaches minus infinity, but make it complex instead of real. In order to keep $-A_1$ above H_1 for the more positive values of V and V', the H_1 bounds are inverted before $-A_1$ is selected, yielding for $-A_1$ the function

$$-A_1 = 4.00 - 20V = 4.00 + I - 20V'$$

The bounds for H_2 are shown plotted in Figure 53. $-A_2$ is chosen to be $5.30 + 40V$ ($= 5.30 + 40V'$). A problem arises here in connection with the H_2^- bound on the V' sheet. If in H_1^- we follow the normal procedure and replace all complex values for V' more negative than -0.3 with minus infinity, the H_2^- bound will be the dotted line in Figure 53, and $-A_2$ will not fall between H_2^+ and H_2^-. Therefore we return to H_1^- and include in our calculations some of the complex values we would have omitted following the normal policy. Carrying forward these points, which are marked with asterisks in the tabulation above, we find the revised H_2^- is such that $-A_2$ now falls between the bounds as desired.

$H_3^{\pm}$ is plotted on Figure 53. A constant is a suitable choice for $-A_3$, and rather than favor either the V or V' curves it is located about midway between the minimum of H_3^+ on V' and the maximum of H_3^- on V, at -4.00. $H_4^{\pm}$ is plotted on Figure 54. It is then inverted so that $-A_4$ will fall above H_4^+ at large values of V, and $-A_4$ is chosen equal to $-4.00 - 20V$, which again is a compromise choice between the best selections on either sheet. $H_5^{\pm}$ is plotted in Figure 54. The cusps in $H_4^{\pm}$ and $H_5^{\pm}$ in the vicinity of V equals 0.1 and V' equals -0.1 result from the sharp breaks in the prescribed function shown in Figure 51 at v equals $\pm$ 0.2. A selection for $-A_5$ of $1.50 + 40V$ falls within the tolerance bounds for all frequencies on both sheets and the expansion is terminated. The desired approximant is

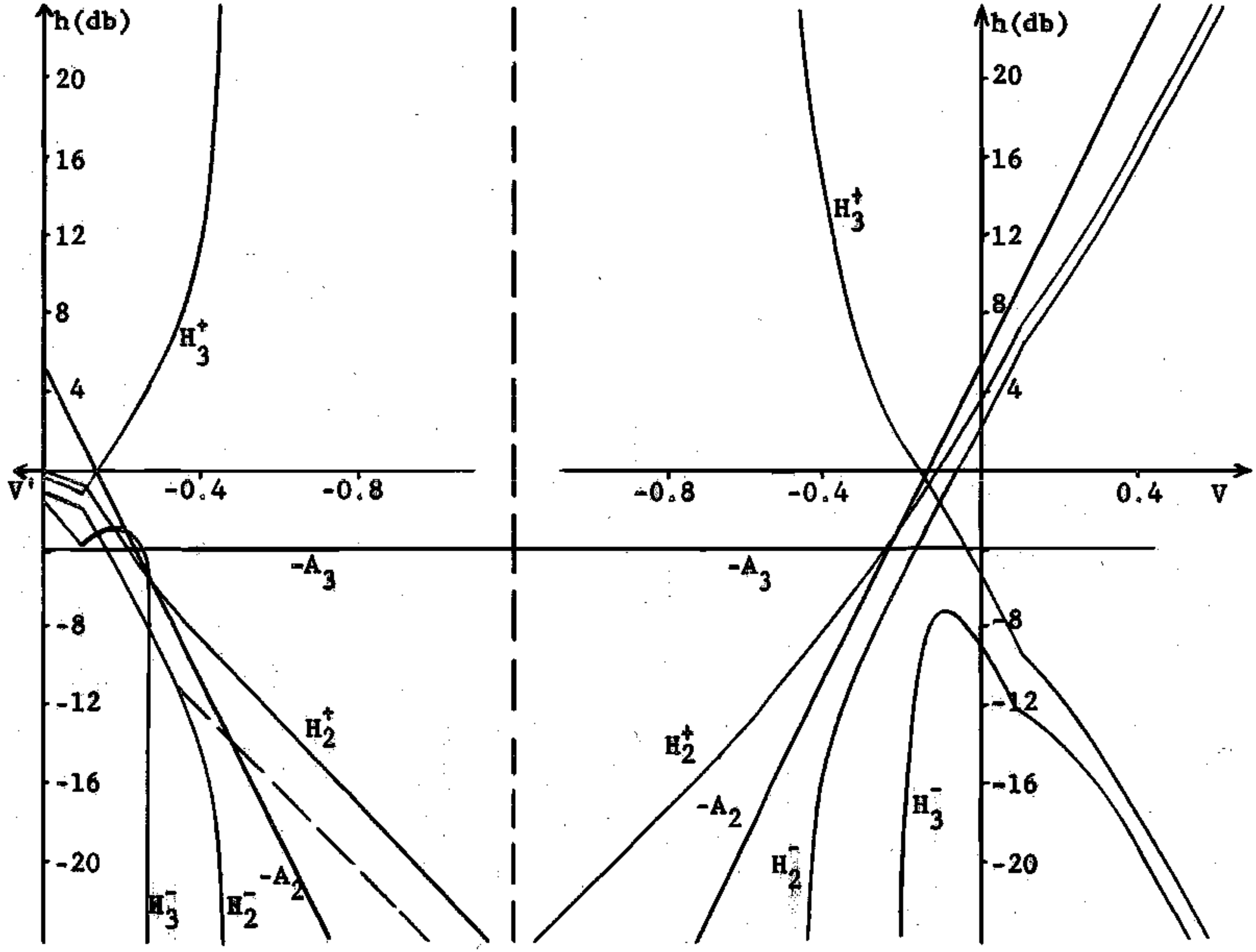

Fig. 53. Fifth Example Continued.

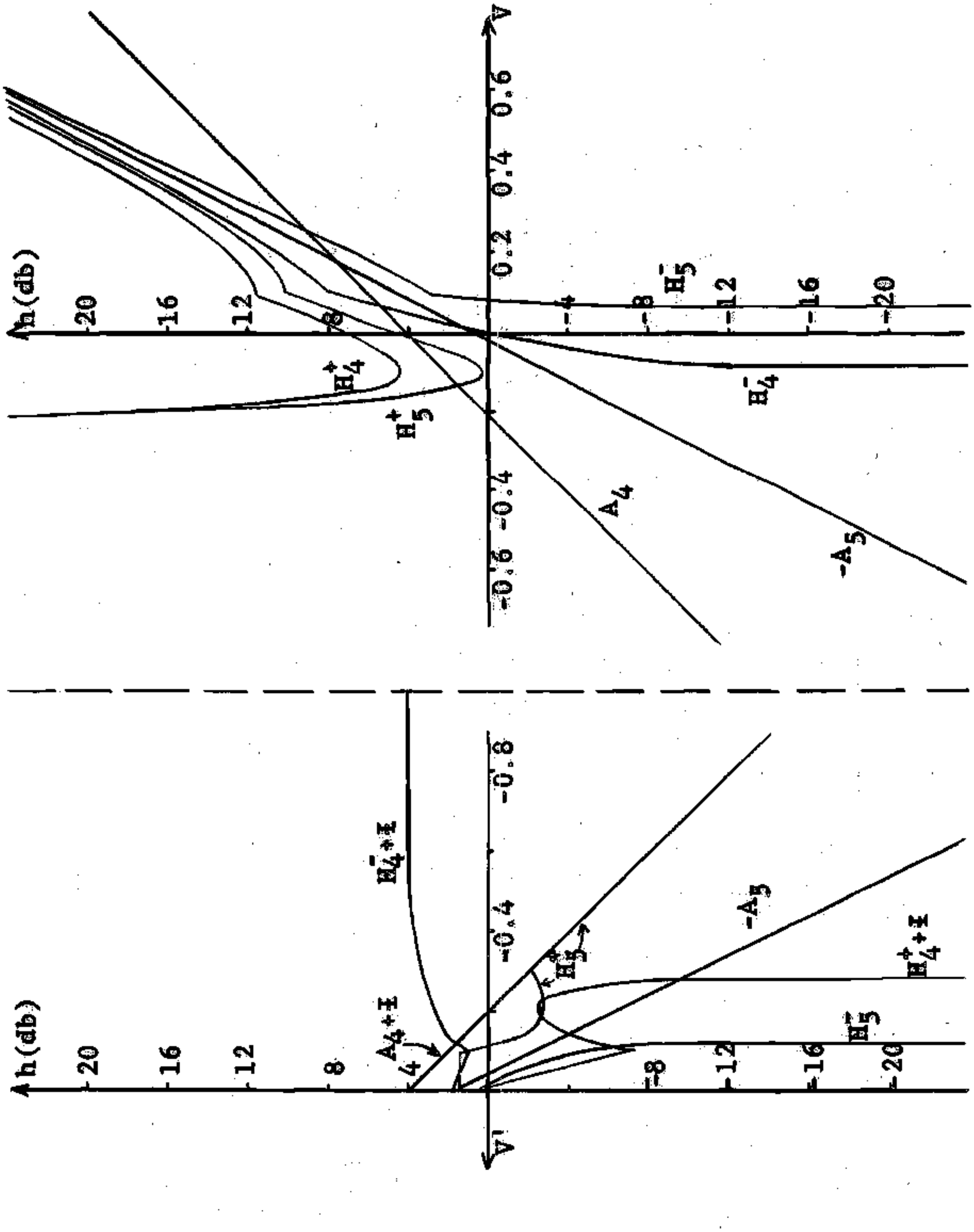

Fig. 54. Fifth Example Continued.

$$G_{o5} = \cfrac{1}{0.39811 + 0.39811\lambda^2 + \cfrac{1}{\cfrac{0.29512}{\lambda^4} + \cfrac{1}{2.5119 + 2.5119\lambda^2 + \cfrac{1}{\cfrac{0.70795}{\lambda^4}}}}}$$

Expressed as a rational fraction,

$$G_{o5} = \frac{1.0031\lambda^4 + 0.52481\lambda^2 + 0.52481}{\lambda^8 + 2.1776\lambda^6 + 2.3866\lambda^4 + 0.41786\lambda^2 + 0.20893}$$

With the substitution of ω^2-1 for λ^2, the approximant is

$$G_{o5} = \frac{1.0031\,\omega^4 - 1.4814\,\omega^2 + 1.0031}{\omega^8 - 1.8224\,\omega^6 + 1.8538\,\omega^4 - 1.8224\,\omega^2 + 1.0001}$$

Sixth example: approximation with functions asymptotic at zero and infinity.--The tolerance bounds of a prescribed function are portrayed in Figure 55. The function is symmetrical about v equals zero, and since in such cases this method will produce elements that are also symmetrical about v equals zero, only half of the points on each remainder need be calculated. The values of the H_o bounds and succeeding remainders for the negative half-axis of v are given in the tabulation below.

Table 8. Values of Quantities Pertaining to Sixth Example

v	H_o^+	H_o^-	$-A_o$	H_1^-	H_1^+
-1.2	-14.00	-14.50	-14.00		4.86
-1.1	-12.00	-12.50	-12.00		2.86
-1.0	-10.00	-10.50	-10.00	$-\infty$	0.86
-0.9	- 8.07	- 8.57	- 8.00	- 9.89	-0.53
-0.8	- 6.16	- 6.66	- 6.00	- 8.26	-1.85
-0.7	- 4.27	- 4.77	- 4.01	- 8.09	-3.17
-0.6	- 2.42	- 2.92	- 2.02	- 8.14	-4.36
-0.5	- 0.60	- 1.10	- 0.04	- 8.57	-5.54
-0.4	- 0.42	- 0.92	1.89	- 3.43	-2.30
-0.3	- 0.27	- 0.77	3.73	- 1.93	-1.13
-0.2	- 0.16	- 0.66	5.36	- 1.27	-0.59
-0.1	- 0.07	- 0.57	6.54	- 1.00	-0.37
0	0	- 0.50	6.99	- 0.97	-0.35

v	H_2^-	H_2^+	$-A_2$	H_3^+	H_3^-
-1.2		4.60	-51.06		
-1.1		2.50	-45.06		
-1.0		0.27	-39.06		
-0.9		- 1.37	-33.06		
-0.8		- 3.03	-27.06		
-0.7		- 4.86	-21.06		
-0.6		- 6.75	-15.06		
-0.5	$-\infty$	- 9.06	- 9.06	$+\infty$	$-\infty$
-0.4	- 5.25	- 3.66	- 3.06	1.23	- 5.23
-0.3	- 3.13	- 2.11	2.94	1.90	0.48
-0.2	- 2.28	- 1.44	8.92	1.94	1.02
-0.1	- 1.94	- 1.17	14.68	1.84	1.06
0	- 1.91	- 1.15	17.93	1.86	1.10

The low-frequency asymptote of H_o^+ is

$$-A_o^o = 10 + 20v$$

and the high-frequency asymptote is

$$-A_o^i = 10 - 20v$$

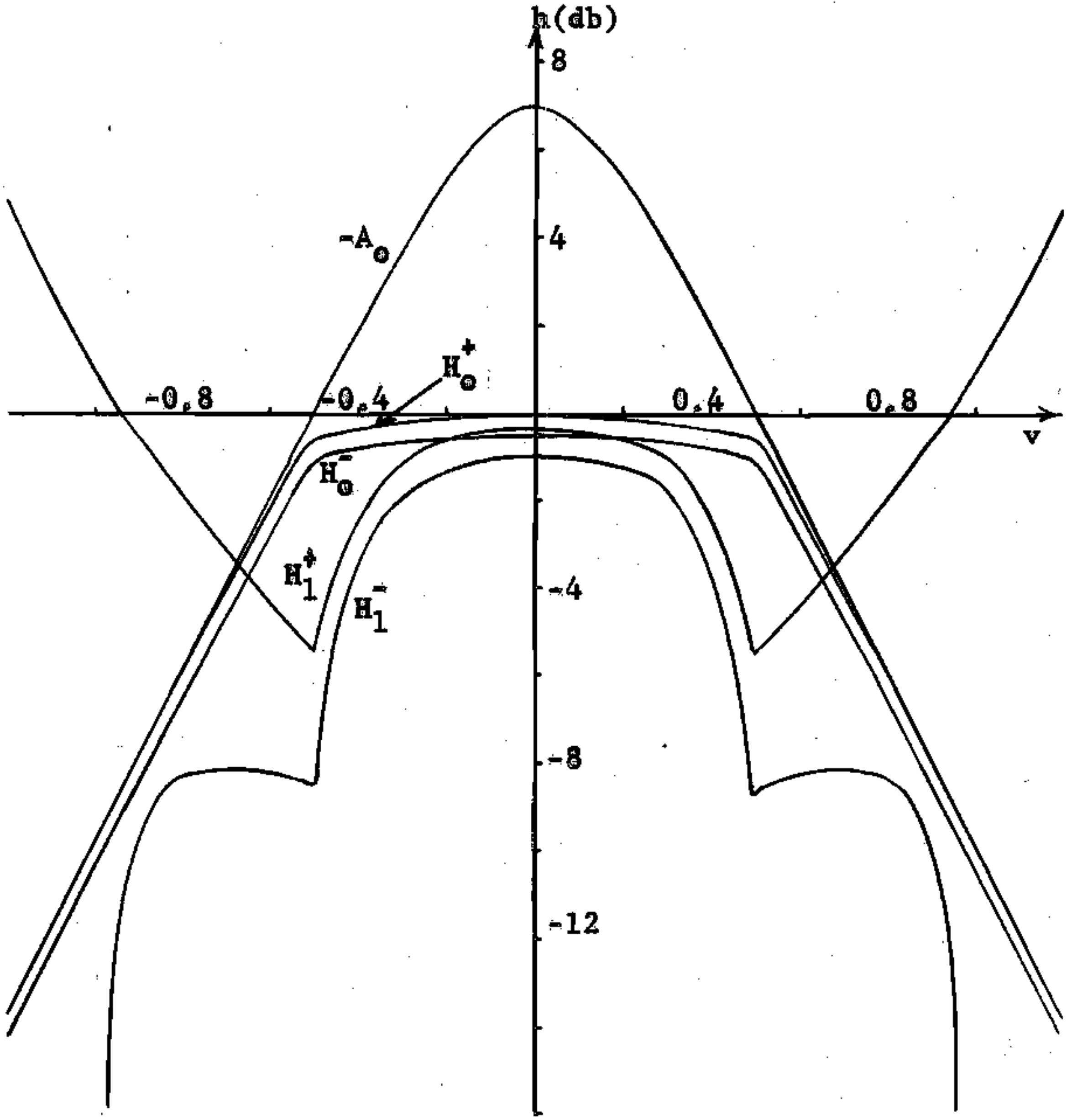

Fig. 55. Sixth Example.

Since the slope p_o^o of the low-frequency asymptote is greater than the slope p_o^i of the high-frequency asymptote, equation (57) is used to calculate the first element.

$$-A_o = -A_o^o - L(\mathrm{I} - A_o^o + A_o^i)$$

$$= 10 + 20v - L(\mathrm{I} + 40v)$$

$$= 10 \log \frac{1}{\dfrac{1}{10\omega^2} + \dfrac{\omega^2}{10}}$$

$$a_o = \frac{\omega^4 + 1}{10\omega^2}$$

The resulting bounds of H_1 are plotted in Figure 55. The shape of H_1^- indicates that a constant would be a suitable choice for $-A_1$. Normally in this method when the high and low-frequency asymptotes are parallel, the formulae for selecting an element asymptotic to both fail, but in this case, since the high and low-frequency asymptotes coincide, the method may be pursued. In order to keep $-A_1$ above the H_1 bounds at center frequencies H_1 is inverted before $-A_1$ is selected. $-A_1$ is made tangent to the bound with the approximately constant shelf in order to stay away from the sharp point in the opposite bound at v equals -0.5. This selection will hold down the order to the final result. Accordingly

$$-A_1 = 8.09$$

$$a_1 = 0.15524$$

The H_2 bounds are plotted in Figure 56. The low-frequency asymptote of H_2 is assumed to pass through -9.06 at v equals -0.5 with a 60 db. per decade slope, which is the smallest slope that it can have without intersecting H_2^-. Then

$$-A_2^o = 20.94 + 60v$$

$$-A_2^i = 20.94 - 60v$$

$$-A_2 = 20.94 + 60v - L(I + 120v)$$

the latter being calculated from equation (57) as was $-A_o$.

$$a_2 = \frac{1}{124.17\omega^6} + \frac{\omega^6}{124.17}$$

The H_3 bounds are also plotted in Figure 56. $-A_3$ is chosen to be 1.18, which falls between the bounds for all frequencies. Thus a_3 equals 0.76208. The desired approximant is

$$G_{o3} = \cfrac{1}{\cfrac{\omega^4 + 1}{10\omega^2} + 0.15524 + \cfrac{1}{\cfrac{\omega^{12} + 1}{124.17\omega^6} + \cfrac{1}{0.76208}}}$$

$$= \frac{10\omega^{14} + 1629.4\omega^8 + 10\omega^2}{\omega^{16} + 1.5524\omega^{14} + \omega^{12} + 162.94\omega^{10} + 1494.6\omega^8 + 162.94\omega^6 + \omega^4 + 1.5524\omega^2 + 1}$$

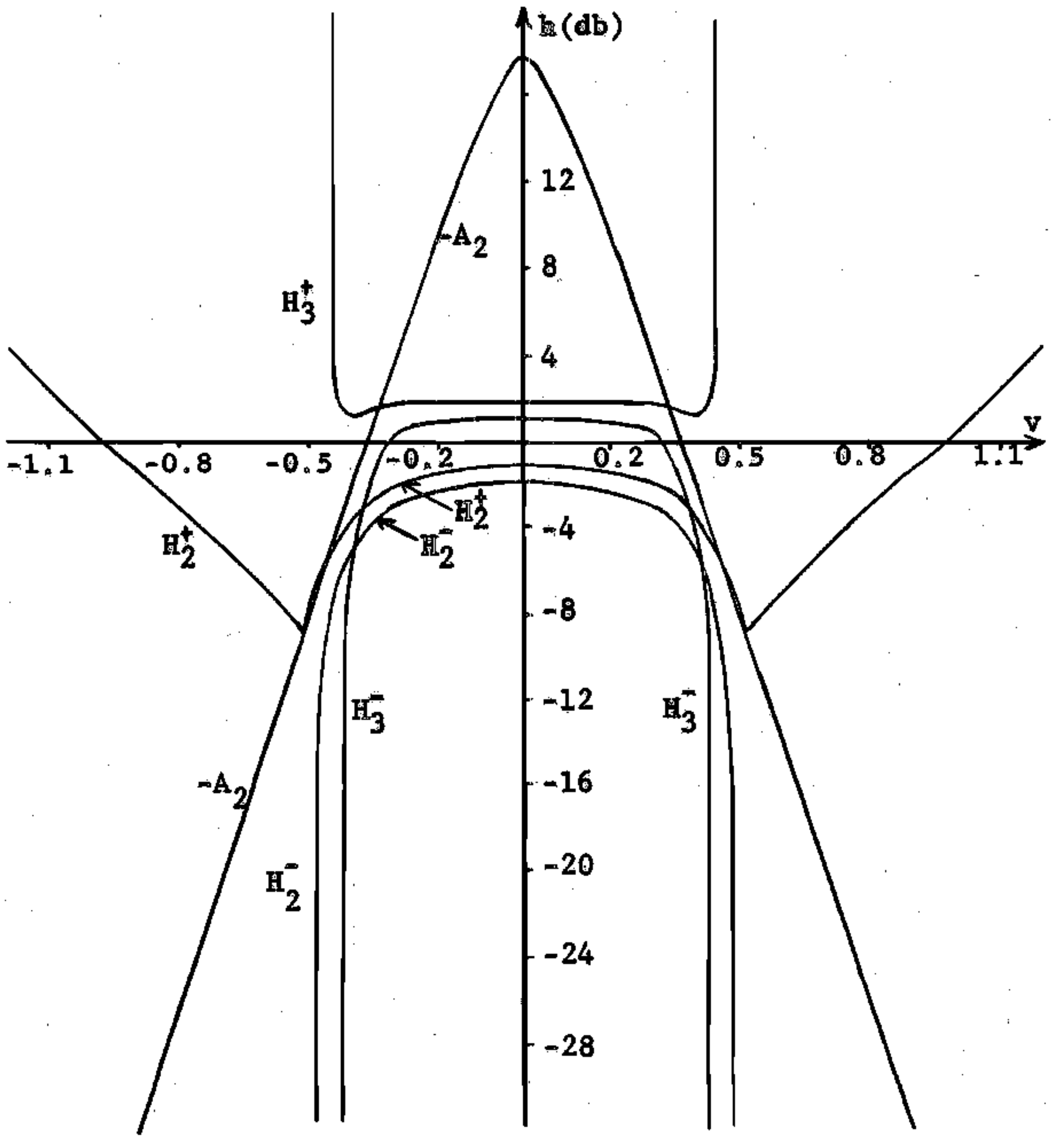

Fig. 56. Sixth Example Continued.

APPENDIX B

TABLES OF STANDARD COMPONENTS

Standard components.--The tables in this appendix give values of functions of the following two types.

$$\text{(I)} \quad 10 \log(1 + \omega^2)$$

$$\text{(II)} \quad 10 \log(1 + 2c\,\omega^2 + \omega^4)$$

These functions are called standard components because the logarithmic expression of any positive rational function of ω^2 is composed of the sum of selected components of the above two forms. The curves are tabulated over a range of three decades at intervals of one-tenth of a decade, with additional values included at $\log \omega = \pm 0.05$, in the region where the curves are generally changing most rapidly. Components of the second form above are tabulated for twenty-one selected values of 2c. The general shape of the standard components is sketched in Figure 57. Graphs are given in many standard texts, but cannot generally be read to enough significant figures for the purposes of this study.

To extend the tabulated range of a standard component or to increase the number of tabulated points in the given range, the L(u) tables of Appendix C may be used in conjunction with the following formulae:

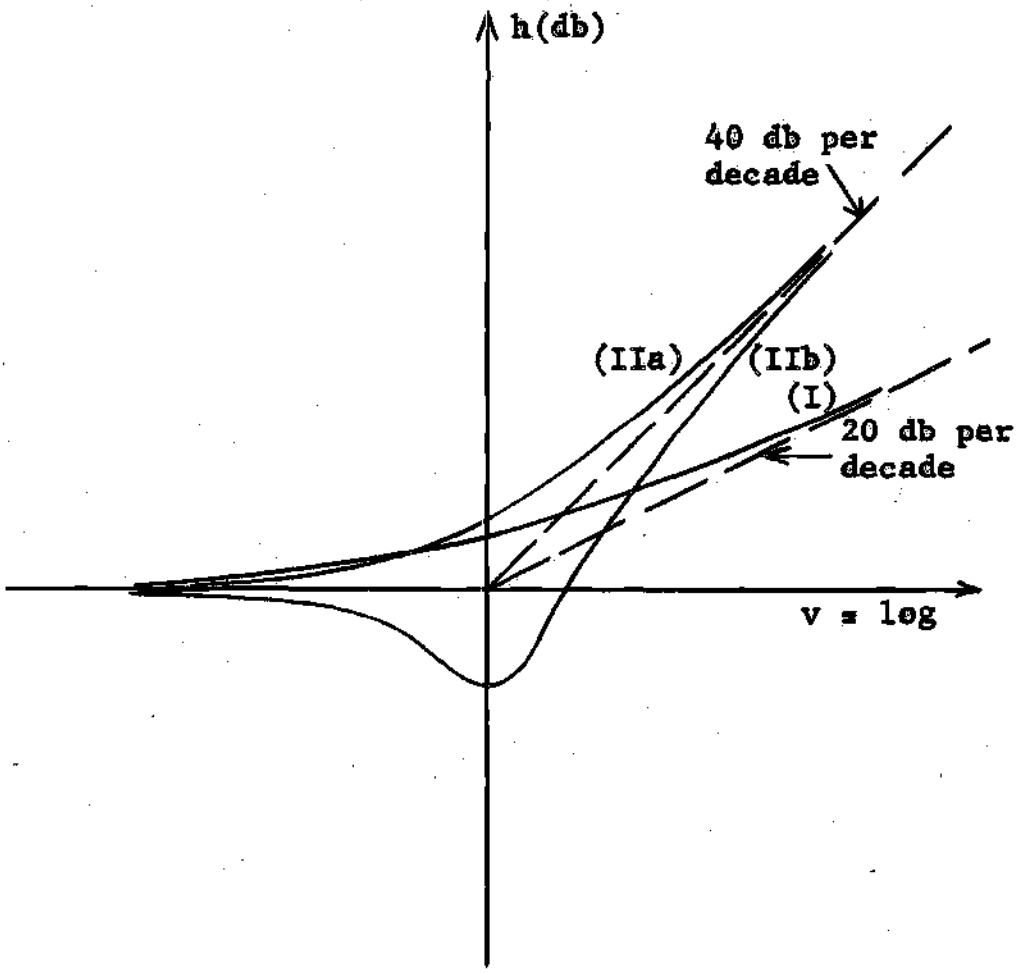

Fig. 57. General Shape of Standard Component Curves.

(I) $10 \log(1 + \omega^2)$

(IIa) $10 \log(1 + 2c\omega^2 + \omega^4)$ for positive c.

(IIb) $10 \log(1 + 2c\omega^2 + \omega^4)$ for negative c.

$$10 \log(1 + \omega^2) = L(20v + I) \tag{68}$$

$$10 \log(1 + 2c\,\omega^2 + \omega^4) = L(40v + I) + $$

$$+ L\left[10 \log 2c + 20v + I - L(40v + I)\right] \tag{69}$$

Tables.--In Table 9 the first column gives values of v (equals log ω); the second gives corresponding values of 10 log$(1 + \omega^2)$; and succeeding columns refer to functions of the type 10 log$(1 + 2c\,\omega^2 + \omega^4)$, with the value of 2c used to identify the column.

Table 9. Standard Components

v	(I)	2.0000	1.1623	0.5119	0	-0.4151	-0.7411
-1.50	0.004	0.009	0.005	0.002	0.000	-0.002	-0.003
-1.40	0.007	0.014	0.008	0.004	0.000	-0.003	-0.005
-1.30	0.011	0.022	0.013	0.006	0.000	-0.005	-0.008
-1.20	0.017	0.034	0.020	0.009	0.000	-0.007	-0.013
-1.10	0.027	0.054	0.032	0.014	0.000	-0.011	-0.020
-1.00	0.043	0.086	0.051	0.023	0.000	-0.018	-0.032
-0.90	0.068	0.136	0.081	0.036	0.001	-0.028	-0.050
-0.80	0.108	0.216	0.128	0.058	0.003	-0.043	-0.079
-0.70	0.170	0.339	0.203	0.095	0.007	-0.065	-0.123
-0.60	0.266	0.531	0.323	0.155	0.017	-0.098	-0.190
-0.50	0.414	0.827	0.516	0.258	0.043	-0.139	-0.288
-0.40	0.639	1.278	0.825	0.439	0.108	-0.180	-0.421
-0.30	0.973	1.947	1.319	0.762	0.266	-0.183	-0.570
-0.20	1.455	2.911	2.098	1.343	0.639	-0.029	-0.637
-0.10	2.124	4.248	3.286	2.358	1.455	0.555	-0.313
-0.05	2.539	5.078	4.072	3.091	2.124	1.144	0.180
0	3.010	6.021	5.000	4.000	3.010	2.000	1.000
0.05	3.539	7.078	6.072	5.091	4.124	3.144	2.180
0.10	4.124	8.248	7.286	6.358	5.455	4.555	3.688
0.20	5.455	10.911	10.089	9.343	8.639	7.971	7.363
0.30	6.973	13.947	13.319	12.762	12.266	11.818	11.430
0.40	8.639	17.278	16.825	16.439	16.108	15.820	15.579
0.50	10.414	20.827	20.516	20.258	20.043	19.861	19.712
0.60	12.266	24.532	24.323	24.155	24.017	23.903	23.810
0.70	14.170	28.339	28.203	28.095	28.007	27.935	27.877
0.80	16.108	32.216	32.128	32.058	32.003	31.957	31.921
0.90	18.068	36.136	36.081	36.036	36.001	35.972	35.950

Table 9. Standard Components (continued)

v	(I)	2.0000	1.1623	0.5119	0	-0.4151	-0.7411
1.00	20.043	40.086	40.051	40.023	40.000	39.982	39.968
1.10	22.027	44.054	44.032	44.014	44.000	43.989	43.980
1.20	24.017	48.034	48.020	48.009	48.000	47.993	47.987
1.30	26.011	52.022	52.013	52.006	52.000	51.995	51.992
1.40	28.007	56.014	56.008	56.004	56.000	55.997	55.995
1.50	30.004	60.009	60.005	60.002	60.000	59.998	59.997

v	-1.0000	-1.2057	-1.3690	-1.5000	-1.6000	-1.6838
-1.50	- 0.004	- 0.005	- 0.006	- 0.007	- 0.007	- 0.007
-1.40	- 0.007	- 0.008	- 0.009	- 0.010	- 0.011	- 0.012
-1.30	- 0.011	- 0.013	- 0.015	- 0.016	- 0.017	- 0.018
-1.20	- 0.017	- 0.021	- 0.024	- 0.026	- 0.028	- 0.029
-1.10	- 0.027	- 0.033	- 0.038	- 0.041	- 0.044	- 0.046
-1.00	- 0.043	- 0.052	- 0.059	- 0.065	- 0.070	- 0.073
-0.90	- 0.068	- 0.083	- 0.094	- 0.103	- 0.111	- 0.116
-0.80	- 0.108	- 0.131	- 0.149	- 0.164	- 0.175	- 0.185
-0.70	- 0.169	- 0.206	- 0.236	- 0.260	- 0.278	- 0.294
-0.60	- 0.265	- 0.325	- 0.374	- 0.413	- 0.443	- 0.469
-0.50	- 0.410	- 0.509	- 0.589	- 0.655	- 0.706	- 0.749
-0.40	- 0.622	- 0.788	- 0.925	- 1.038	- 1.126	- 1.202
-0.30	- 0.905	- 1.191	- 1.431	- 1.635	- 1.797	- 1.937
-0.20	- 1.190	- 1.685	- 2.122	- 2.508	- 2.827	- 3.114
-0.10	- 1.151	- 1.956	- 2.722	- 3.452	- 4.105	- 4.740
-0.05	- 0.775	- 1.718	- 2.648	- 3.571	- 4.436	- 5.325
0	0	- 1.000	- 2.000	- 3.010	- 3.979	- 5.000
0.05	1.225	0.282	- 0.648	- 1.571	- 2.436	- 3.325
0.10	2.849	2.044	1.278	0.548	- 0.105	- 0.740
0.20	6.810	6.315	5.878	5.492	5.173	4.886
0.30	11.095	10.810	10.569	10.365	10.203	10.063
0.40	15.378	15.212	15.075	14.962	14.874	14.798
0.50	19.590	19.491	19.411	19.345	19.294	19.251
0.60	23.735	23.675	23.627	23.587	23.557	23.532
0.70	27.831	27.794	27.764	27.740	27.722	27.706
0.80	31.892	31.869	31.851	31.836	31.825	31.815
0.90	35.932	35.917	35.906	35.897	35.889	35.884
1.00	39.957	39.948	39.941	39.935	39.930	39.927
1.10	43.973	43.967	43.962	43.959	43.956	43.954
1.20	47.983	47.979	47.976	47.974	47.972	47.971
1.30	51.989	51.987	51.985	51.984	51.983	51.982
1.40	55.993	55.992	55.991	55.990	55.989	55.988
1.50	59.996	59.995	59.994	59.993	59.993	59.993

Table 9. Standard Components (continued)

v	-1.7488	-1.8000	-1.8415	-1.8471	-1.9000	-1.9206
-1.50	- 0.008	- 0.008	- 0.008	- 0.008	- 0.008	- 0.008
-1.40	- 0.012	- 0.012	- 0.013	- 0.013	- 0.013	- 0.013
-1.30	- 0.019	- 0.020	- 0.020	- 0.020	- 0.021	- 0.021
-1.20	- 0.030	- 0.031	- 0.032	- 0.033	- 0.033	- 0.033
-1.10	- 0.048	- 0.050	- 0.051	- 0.052	- 0.052	- 0.053
-1.00	- 0.076	- 0.078	- 0.081	- 0.082	- 0.083	- 0.084
-0.90	- 0.121	- 0.125	- 0.128	- 0.130	- 0.132	- 0.133
-0.80	- 0.192	- 0.198	- 0.203	- 0.207	- 0.210	- 0.212
-0.70	- 0.306	- 0.316	- 0.323	- 0.329	- 0.334	- 0.338
-0.60	- 0.488	- 0.504	- 0.517	- 0.527	- 0.535	- 0.541
-0.50	- 0.783	- 0.809	- 0.831	- 0.848	- 0.862	- 0.873
-0.40	- 1.261	- 1.309	- 1.347	- 1.378	- 1.403	- 1.422
-0.30	- 2.049	- 2.140	- 2.215	- 2.274	- 2.322	- 2.361
-0.20	- 3.351	- 3.547	- 3.712	- 3.847	- 3.957	- 4.047
-0.10	- 5.306	- 5.810	- 6.267	- 6.663	- 7.005	- 7.298
-0.05	- 6.164	- 6.965	- 7.742	- 8.468	- 9.145	- 9.772
0	- 6.000	- 6.990	- 8.000	- 9.000	-10.000	-11.000
0.05	- 4.164	- 4.965	- 5.742	- 6.468	- 7.145	- 7.772
0.10	- 1.306	- 1.810	- 2.267	- 2.663	- 3.005	- 3.298
0.20	4.649	4.453	4.288	4.153	4.043	3.953
0.30	9.951	9.860	9.785	9.726	9.678	9.639
0.40	14.739	14.691	14.653	14.622	14.597	14.578
0.50	19.218	19.191	19.169	19.152	19.138	19.127
0.60	23.512	23.496	23.483	23.473	23.465	23.459
0.70	27.694	27.685	27.677	27.671	26.666	27.662
0.80	31.808	31.802	31.797	31.794	31.791	31.788
0.90	35.879	35.875	35.872	35.870	35.868	35.867
1.00	39.924	39.922	39.919	39.918	39.917	39.916
1.10	43.952	43.950	43.949	43.948	43.948	43.947
1.20	47.970	47.969	47.968	47.967	47.967	47.967
1.30	51.981	51.980	51.980	51.980	51.979	51.979
1.40	55.988	55.988	55.987	55.987	55.987	55.987
1.50	59.992	59.992	59.992	59.992	59.992	59.992

Table 9. Standard Components (continued)

v	-1.9369	-1.9500	-1.9600
-1.50	- 0.008	- 0.008	- 0.009
-1.40	- 0.013	- 0.013	- 0.014
-1.30	- 0.021	- 0.021	- 0.021
-1.20	- 0.034	- 0.034	- 0.034
-1.10	- 0.053	- 0.054	- 0.054
-1.00	- 0.085	- 0.085	- 0.086
-0.90	- 0.134	- 0.135	- 0.136
-0.80	- 0.214	- 0.215	- 0.216
-0.70	- 0.341	- 0.344	- 0.345
-0.60	- 0.546	- 0.550	- 0.554
-0.50	- 0.881	- 0.888	- 0.894
-0.40	- 1.434	- 1.451	- 1.460
-0.30	- 2.392	- 2.416	- 2.435
-0.20	- 4.119	- 4.177	- 4.223
-0.10	- 7.545	- 7.752	- 7.920
-0.05	-10.341	-10.860	-11.304
0	-12.000	-13.010	-13.979
0.05	- 8.341	- 8.860	- 9.304
0.10	- 3.545	- 3.752	- 3.920
0.20	3.882	3.823	3.777
0.30	9.609	9.584	9.565
0.40	14.562	14.550	14.540
0.50	19.119	19.112	19.106
0.60	23.454	23.450	23.446
0.70	27.659	27.657	27.655
0.80	31.786	31.785	31.784
0.90	35.866	35.865	35.864
1.00	39.915	39.915	39.914
1.10	43.947	43.946	43.946
1.20	47.966	47.966	47.966
1.30	51.979	51.979	51.979
1.40	55.987	55.987	55.986
1.50	59.992	59.992	59.991

APPENDIX C

PROPERTIES AND TABLES OF L(u)

Definition.--The basic recurrence formula in the continued fraction expansion of a prescribed function relates each remainder to the preceding remainder in the following manner.

$$G_{k+1} = \frac{1}{G_k} - a_k$$

In performing the calculations in the logarithmic domain the following equation arise.

$$10 \log G_{k+1} = 10 \log \frac{1}{G_k} + 10 \log(1 - a_k G_k)$$

$$H_{k+1} = - H_k + 10 \log(1 - \text{antilog}\frac{A_k + H_k}{10})$$

The second term of the right-hand part of the last equation is a function of a single variable, $A_k + H_k$. In order to avoid the necessity for repetitive performance of the three-step calculation indicated by this term, it is designated a function and tabulated in this appendix. Let

$$L(u) = 10 \log(1 - 10^{\frac{u}{10}}) \tag{70}$$

Units and complex values.--The units of u and L(u) are decibels.

It frequently occurs that u or L(u) represents the decible equivalent of a negative number. To handle these expeditiously the symbol $\bar{I}$ is assigned to 10 log(-1). Where 10 log b equals B, 10 log(-b) equals B + $\bar{I}$. It follows that

$$L(u + \bar{I}) = 10 \log(1 - 10^{\frac{u+\bar{I}}{10}}) = 10 \log(1 + 10^{\frac{u}{10}}) \qquad (71)$$

Properties of L(u).--The following properties of L(u) are readily derived from the definition (70) of the function.

$$L\left\{L(u)\right\} = u \qquad (72)$$

$$L(-u) = L(u) - u + \bar{I} \qquad (73)$$

$$\frac{dL(u)}{du} = \frac{1}{1 - 10^{\frac{-u}{10}}} \qquad (74)$$

$$L(u + w) = L(u) + L\left\{L(w) - L(-u)\right\} \qquad (75)$$

Other relations developed from the above are:

$$L(-u + \bar{I}) = L(u + \bar{I}) - u \qquad (76)$$

$$L\left\{-L(u)\right\} = - L(-u) \qquad (77)$$

$$L(2u) = L(u) + L(u + \bar{I}) \qquad (78)$$

The L-functions are related to the hyperbolic functions as follows:

$$L(u) = \bar{I} + \frac{u}{2} + 10 \log 2 + 10 \log \sinh(\frac{u}{20} \ln 10) \qquad (79)$$

$$L(u + I) = \frac{u}{2} + 10 \log 2 + 10 \log \cosh(\frac{u}{20} \ln 10) \qquad (80)$$

Extension of the tables.--The tables of $L(u)$ presented in the next section give the values of $L(u)$ for values of u at increments of 0.01, from 0.01 to 30.00. The tables are intended primarily to find $L(u)$ to the nearest hundredth of a decibel. Consequently the third decimal figure given is used normally to round off the second decimal. The third decimal place is carried, however, so that greater accuracy may be achieved if desired. In order to obtain this accuracy entry into the tables must be made with three-decimal figures and interpolation is involved. Linear interpolation is satisfactory except for real u's with magnitudes less than unity. Extension and interpolation formulae are given below.

(1) As $u \longrightarrow \infty$:

$$L(u) \longrightarrow I + u - 4.343(10^{\frac{-u}{10}}) \qquad (81)$$

$$L(-u) \longrightarrow - 4.343(10^{\frac{-u}{10}}) \qquad (82)$$

$$L(u + I) \longrightarrow u + 4.343(10^{\frac{-u}{10}}) \qquad (83)$$

$$L(-u + I) \longrightarrow 4.343(10^{\frac{-u}{10}}) \qquad (84)$$

For $u \geq 30$, the error introduced by using the expressions above is less than 0.00001.

(2) As $u \longrightarrow 0$:

$$L(u) \longrightarrow I + \frac{u}{2} - 6.377843 + 10 \log u \qquad (85)$$

$$L(-u) \longrightarrow -\frac{u}{2} - 6.377843 + 10 \log u \qquad (86)$$

$$L(u + \mathrm{I}) \longrightarrow \frac{u}{2} + 3.010300 \qquad (87)$$

$$L(-u + \mathrm{I}) \longrightarrow -\frac{u}{2} + 3.010300 \qquad (88)$$

For $u \leqq 0.01$, the error introduced is less than 0.0001.

(3) Interpolation.

(a) For all complex u, and for real u where $|u| \geqq 1$, linear interpolation gives accuracy to three decimal places.

(b) For $|u| \leqq 0.07$, equations (85) and (86) give accuracy to three decimal places.

(c) For $0.07 < |u| < 1.0$, the following extensions of (85) and (86) must be used to obtain three-decimal accuracy:

$$L(u) \approx \mathrm{I} + \frac{u}{2} - 6.377843 + 10 \log u + 0.096u^2 \qquad (89)$$

$$L(-u) \approx -\frac{u}{2} - 6.377843 + 10 \log u + 0.096u^2 \qquad (90)$$

Tables.--The tables are entered with a real positive value of u. Columns are provided for L(u), L(-u), L(u+I), and L(-u+I), the proper column to be selected depending upon the sign and complex nature of the given variable. It is important to note that L(u) is complex for real positive u; since only the real part is given in the tables, that column is headed "L(u) + I". On the other hand if the variable is negative real or complex, the L-function is real.

Table 10. Tables of L(u).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
0.01	-26.373	-26.383	3.015	3.005
0.02	-23.358	-23.378	3.020	3.000
0.03	-21.592	-21.622	3.025	2.995
0.04	-20.337	-20.377	3.030	2.990
0.05	-19.363	-19.413	3.035	2.985
0.06	-18.566	-18.626	3.040	2.980
0.07	-17.892	-17.962	3.045	2.975
0.08	-17.307	-17.387	3.050	2.970
0.09	-16.790	-16.880	3.056	2.966
0.10	-16.328	-16.428	3.061	2.961
0.11	-15.909	-16.019	3.066	2.956
0.12	-15.526	-15.646	3.071	2.951
0.13	-15.173	-15.303	3.076	2.946
0.14	-14.846	-14.986	3.081	2.941
0.15	-14.542	-14.692	3.086	2.936
0.16	-14.256	-14.416	3.091	2.931
0.17	-13.988	-14.158	3.096	2.926
0.18	-13.7348	-13.9148	3.101	2.921
0.19	-13.4949	-13.6849	3.106	2.916
0.20	-13.267	-13.467	3.111	2.911
0.21	-13.050	-13.260	3.117	2.907
0.22	-12.843	-13.063	3.122	2.902
0.23	-12.645	-12.875	3.127	2.897
0.24	-12.455	-12.695	3.132	2.892
0.25	-12.273	-12.523	3.137	2.887
0.26	-12.097	-12.357	3.142	2.882
0.27	-11.929	-12.199	3.147	2.877
0.28	-11.766	-12.046	3.153	2.873
0.29	-11.608	-11.898	3.158	2.868
0.30	-11.456	-11.756	3.163	2.863
0.31	-11.308	-11.618	3.168	2.858
0.32	-11.165	-11.485	3.173	2.853
0.33	-11.027	-11.357	3.178	2.848
0.34	-10.892	-11.232	3.184	2.844
0.35	-10.761	-11.111	3.189	2.839
0.36	-10.634	-10.994	3.194	2.834
0.37	-10.510	-10.880	3.199	2.829
0.38	-10.389	-10.769	3.204	2.824
0.39	-10.271	-10.661	3.210	2.820
0.40	-10.156	-10.556	3.2149	2.8149

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
0.41	-10.043	-10.453	3.220	2.810
0.42	- 9.934	-10.354	3.225	2.805
0.43	- 9.826	-10.256	3.231	2.801
0.44	- 9.721	-10.161	3.236	2.796
0.45	- 9.619	-10.069	3.241	2.791
0.46	- 9.518	- 9.978	3.246	2.786
0.47	- 9.420	- 9.890	3.252	2.782
0.48	- 9.323	- 9.803	3.257	2.777
0.49	- 9.229	- 9.719	3.262	2.772
0.50	- 9.136	- 9.636	3.267	2.767
0.51	- 9.0446	- 9.5546	3.273	2.763
0.52	- 8.955	- 9.475	3.278	2.758
0.53	- 8.867	- 9.397	3.283	2.753
0.54	- 8.781	- 9.321	3.289	2.749
0.55	- 8.696	- 9.246	3.294	2.744
0.56	- 8.613	- 9.173	3.299	2.739
0.57	- 8.531	- 9.101	3.3046	2.7346
0.58	- 8.450	- 9.030	3.310	2.730
0.59	- 8.371	- 8.961	3.315	2.725
0.60	- 8.293	- 8.893	3.321	2.721
0.61	- 8.216	- 8.826	3.326	2.716
0.62	- 8.140	- 8.760	3.331	2.711
0.63	- 8.066	- 8.696	3.337	2.707
0.64	- 7.992	- 8.632	3.342	2.702
0.65	- 7.920	- 8.570	3.347	2.697
0.66	- 7.848	- 8.508	3.353	2.693
0.67	- 7.778	- 8.448	3.358	2.688
0.68	- 7.708	- 8.388	3.364	2.684
0.69	- 7.640	- 8.330	3.369	2.679
0.70	- 7.572	- 8.272	3.374	2.674
0.71	- 7.505	- 8.215	3.380	2.670
0.72	- 7.440	- 8.160	3.385	2.665
0.73	- 7.3745	- 8.1045	3.391	2.661
0.74	- 7.310	- 8.050	3.396	2.656
0.75	- 7.247	- 7.997	3.401	2.651
0.76	- 7.184	- 7.944	3.407	2.647
0.77	- 7.122	- 7.892	3.412	2.642
0.78	- 7.061	- 7.841	3.418	2.638
0.79	- 7.001	- 7.791	3.423	2.633
0.80	- 6.941	- 7.741	3.429	2.629

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
0.81	- 6.882	- 7.692	3.434	2.624
0.82	- 6.823	- 7.643	3.440	2.620
0.83	- 6.765	- 7.595	3.445	2.615
0.84	- 6.708	- 7.548	3.451	2.611
0.85	- 6.652	- 7.502	3.456	2.606
0.86	- 6.596	- 7.456	3.462	2.602
0.87	- 6.540	- 7.410	3.467	2.597
0.88	- 6.486	- 7.366	3.473	2.593
0.89	- 6.431	- 7.321	3.478	2.588
0.90	- 6.378	- 7.278	3.484	2.584
0.91	- 6.324	- 7.234	3.489	2.579
0.92	- 6.272	- 7.192	3.4946	2.5746
0.93	- 6.220	- 7.150	3.500	2.570
0.94	- 6.168	- 7.108	3.506	2.566
0.95	- 6.117	- 7.067	3.511	2.561
0.96	- 6.066	- 7.026	3.517	2.557
0.97	- 6.016	- 6.986	3.522	2.552
0.98	- 5.966	- 6.946	3.528	2.548
0.99	- 5.917	- 6.907	3.533	2.543
1.00	- 5.868	- 6.868	3.539	2.539
1.01	- 5.820	- 6.830	3.5445	2.5345
1.02	- 5.772	- 6.792	3.550	2.530
1.03	- 5.724	- 6.754	3.556	2.526
1.04	- 5.677	- 6.717	3.561	2.521
1.05	- 5.630	- 6.680	3.567	2.517
1.06	- 5.584	- 6.644	3.573	2.513
1.07	- 5.538	- 6.608	3.578	2.508
1.08	- 5.492	- 6.572	3.584	2.504
1.09	- 5.447	- 6.537	3.589	2.499
1.10	- 5.402	- 6.502	3.595	2.495
1.11	- 5.358	- 6.468	3.601	2.491
1.12	- 5.314	- 6.434	3.606	2.486
1.13	- 5.270	- 6.400	3.612	2.482
1.14	- 5.226	- 6.366	3.618	2.478
1.15	- 5.183	- 6.333	3.623	2.473
1.16	- 5.140	- 6.300	3.629	2.469
1.17	- 5.098	- 6.268	3.6345	2.4645
1.18	- 5.056	- 6.236	3.640	2.460
1.19	- 5.014	- 6.204	3.646	2.456
1.20	- 4.972	- 6.172	3.652	2.452

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
1.21	- 4.931	- 6.141	3.657	2.447
1.22	- 4.890	- 6.110	3.663	2.443
1.23	- 4.849	- 6.079	3.669	2.439
1.24	- 4.809	- 6.049	3.674	2.434
1.25	- 4.769	- 6.019	3.680	2.430
1.26	- 4.729	- 5.989	3.686	2.426
1.27	- 4.689	- 5.959	3.692	2.422
1.28	- 4.650	- 5.930	3.697	2.417
1.29	- 4.611	- 5.901	3.703	2.413
1.30	- 4.572	- 5.872	3.709	2.409
1.31	- 4.534	- 5.844	3.7145	2.4045
1.32	- 4.495	- 5.815	3.720	2.400
1.33	- 4.457	- 5.787	3.726	2.396
1.34	- 4.420	- 5.760	3.732	2.392
1.35	- 4.382	- 5.732	3.738	2.388
1.36	- 4.3447	- 5.7047	3.743	2.383
1.37	- 4.308	- 5.678	3.749	2.379
1.38	- 4.271	- 5.651	3.7548	2.3748
1.39	- 4.234	- 5.624	3.761	2.371
1.40	- 4.198	- 5.598	3.766	2.366
1.41	- 4.162	- 5.572	3.772	2.362
1.42	- 4.126	- 5.546	3.778	2.358
1.43	- 4.090	- 5.520	3.784	2.354
1.44	- 4.054	- 5.494	3.790	2.350
1.45	- 4.019	- 5.469	3.796	2.346
1.46	- 3.984	- 5.444	3.801	2.341
1.47	- 3.949	- 5.419	3.807	2.337
1.48	- 3.914	- 5.394	3.813	2.333
1.49	- 3.880	- 5.370	3.819	2.329
1.50	- 3.845	- 5.345	3.8247	2.3247
1.51	- 3.811	- 5.321	3.831	2.321
1.52	- 3.777	- 5.297	3.836	2.316
1.53	- 3.743	- 5.273	3.842	2.312
1.54	- 3.710	- 5.250	3.848	2.308
1.55	- 3.677	- 5.227	3.854	2.304
1.56	- 3.643	- 5.203	3.860	2.300
1.57	- 3.610	- 5.180	3.866	2.296
1.58	- 3.577	- 5.157	3.872	2.292
1.59	- 3.5446	- 5.1346	3.878	2.288
1.60	- 3.512	- 5.112	3.884	2.284

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
1.61	- 3.480	- 5.090	3.889	2.279
1.62	- 3.448	- 5.068	3.895	2.275
1.63	- 3.416	- 5.046	3.901	2.271
1.64	- 3.384	- 5.024	3.907	2.267
1.65	- 3.352	- 5.002	3.913	2.263
1.66	- 3.320	- 4.980	3.919	2.259
1.67	- 3.289	- 4.959	3.925	2.255
1.68	- 3.258	- 4.938	3.931	2.251
1.69	- 3.227	- 4.917	3.937	2.247
1.70	- 3.196	- 4.896	3.943	2.243
1.71	- 3.165	- 4.875	3.949	2.239
1.72	- 3.134	- 4.854	3.9548	2.2348
1.73	- 3.104	- 4.834	3.961	2.231
1.74	- 3.073	- 4.813	3.967	2.227
1.75	- 3.043	- 4.793	3.973	2.223
1.76	- 3.013	- 4.773	3.979	2.219
1.77	- 2.983	- 4.753	3.9848	2.2148
1.78	- 2.953	- 4.733	3.991	2.211
1.79	- 2.924	- 4.714	3.997	2.207
1.80	- 2.894	- 4.694	4.003	2.203
1.81	- 2.8646	- 4.6746	4.009	2.199
1.82	- 2.835	- 4.655	4.0149	2.1949
1.83	- 2.806	- 4.636	4.021	2.191
1.84	- 2.777	- 4.617	4.027	2.187
1.85	- 2.748	- 4.598	4.033	2.183
1.86	- 2.720	- 4.580	4.039	2.179
1.87	- 2.691	- 4.561	4.045	2.175
1.88	- 2.662	- 4.542	4.051	2.171
1.89	- 2.634	- 4.524	4.057	2.167
1.90	- 2.606	- 4.506	4.063	2.163
1.91	- 2.578	- 4.488	4.069	2.159
1.92	- 2.550	- 4.470	4.076	2.156
1.93	- 2.522	- 4.452	4.082	2.152
1.94	- 2.494	- 4.434	4.088	2.148
1.95	- 2.466	- 4.416	4.094	2.144
1.96	- 2.438	- 4.398	4.100	2.140
1.97	- 2.411	- 4.381	4.106	2.136
1.98	- 2.384	- 4.364	4.112	2.132
1.99	- 2.356	- 4.346	4.118	2.128
2.00	- 2.329	- 4.329	4.124	2.124

Table 10. Tables of L(u)(continued).

u	L(u) + Ī	L(-u)	L(u+Ī)	L(-u+Ī)
2.01	- 2.302	- 4.312	4.131	2.121
2.02	- 2.275	- 4.295	4.137	2.117
2.03	- 2.248	- 4.278	4.143	2.113
2.04	- 2.222	- 4.262	4.149	2.109
2.05	- 2.195	- 4.245	4.155	2.105
2.06	- 2.169	- 4.229	4.161	2.101
2.07	- 2.142	- 4.212	4.167	2.097
2.08	- 2.116	- 4.196	4.174	2.094
2.09	- 2.090	- 4.180	4.180	2.090
2.10	- 2.063	- 4.163	4.186	2.086
2.11	- 2.037	- 4.147	4.192	2.082
2.12	- 2.011	- 4.131	4.198	2.078
2.13	- 1.986	- 4.116	4.2045	2.0745
2.14	- 1.960	- 4.100	4.211	2.071
2.15	- 1.934	- 4.084	4.217	2.067
2.16	- 1.909	- 4.069	4.223	2.063
2.17	- 1.883	- 4.053	4.229	2.059
2.18	- 1.858	- 4.038	4.236	2.056
2.19	- 1.832	- 4.022	4.242	2.052
2.20	- 1.807	- 4.007	4.248	2.048
2.21	- 1.782	- 3.992	4.254	2.044
2.22	- 1.757	- 3.977	4.261	2.041
2.23	- 1.732	- 3.962	4.267	2.037
2.24	- 1.707	- 3.947	4.273	2.033
2.25	- 1.683	- 3.933	4.279	2.029
2.26	- 1.658	- 3.918	4.286	2.026
2.27	- 1.633	- 3.903	4.292	2.022
2.28	- 1.609	- 3.889	4.298	2.018
2.29	- 1.584	- 3.874	4.3045	2.0145
2.30	- 1.560	- 3.860	4.311	2.011
2.31	- 1.536	- 3.846	4.317	2.007
2.32	- 1.511	- 3.831	4.323	2.003
2.33	- 1.487	- 3.817	4.330	2.000
2.34	- 1.463	- 3.803	4.336	1.996
2.35	- 1.439	- 3.789	4.342	1.992
2.36	- 1.415	- 3.775	4.349	1.989
2.37	- 1.392	- 3.762	4.355	1.985
2.38	- 1.368	- 3.748	4.361	1.981
2.39	- 1.344	- 3.734	4.368	1.978
2.40	- 1.321	- 3.721	4.374	1.974

Table 10. Tables of L(u)(continued).

u	L(u) + Φ	L(-u)	L(u+Φ)	L(-u+Φ)
2.41	- 1.297	- 3.707	4.380	1.970
2.42	- 1.274	- 3.694	4.387	1.967
2.43	- 1.250	- 3.680	4.393	1.963
2.44	- 1.227	- 3.667	4.399	1.959
2.45	- 1.204	- 3.654	4.406	1.956
2.46	- 1.181	- 3.641	4.412	1.952
2.47	- 1.157	- 3.627	4.419	1.949
2.48	- 1.134	- 3.614	4.4249	1.9449
2.49	- 1.112	- 3.602	4.431	1.941
2.50	- 1.089	- 3.589	4.438	1.938
2.51	- 1.066	- 3.576	4.444	1.934
2.52	- 1.043	- 3.563	4.451	1.931
2.53	- 1.020	- 3.550	4.457	1.927
2.54	- 0.998	- 3.538	4.463	1.923
2.55	- 0.975	- 3.525	4.470	1.920
2.56	- 0.953	- 3.513	4.476	1.916
2.57	- 0.930	- 3.500	4.483	1.913
2.58	- 0.908	- 3.488	4.489	1.909
2.59	- 0.886	- 3.476	4.496	1.906
2.60	- 0.863	- 3.463	4.502	1.902
2.61	- 0.841	- 3.451	4.508	1.898
2.62	- 0.819	- 3.439	4.5149	1.8949
2.63	- 0.797	- 3.427	4.521	1.891
2.64	- 0.775	- 3.415	4.528	1.888
2.65	- 0.753	- 3.403	4.534	1.884
2.66	- 0.731	- 3.391	4.541	1.881
2.67	- 0.710	- 3.380	4.547	1.877
2.68	- 0.688	- 3.368	4.554	1.874
2.69	- 0.666	- 3.356	4.560	1.870
2.70	- 0.644	- 3.344	4.567	1.867
2.71	- 0.623	- 3.333	4.573	1.863
2.72	- 0.601	- 3.321	4.580	1.860
2.73	- 0.580	- 3.310	4.586	1.856
2.74	- 0.559	- 3.299	4.593	1.853
2.75	- 0.537	- 3.287	4.599	1.849
2.76	- 0.516	- 3.276	4.606	1.846
2.77	- 0.4946	- 3.2646	4.612	1.842
2.78	- 0.473	- 3.253	4.619	1.839
2.79	- 0.452	- 3.242	4.626	1.836
2.80	- 0.431	- 3.231	4.632	1.832

Table 10. Tables of L(u) (continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
2.81	- 0.410	- 3.220	4.639	1.829
2.82	- 0.389	- 3.209	4.645	1.825
2.83	- 0.368	- 3.198	4.652	1.822
2.84	- 0.348	- 3.188	4.658	1.818
2.85	- 0.327	- 3.177	4.665	1.815
2.86	- 0.306	- 3.166	4.672	1.812
2.87	- 0.285	- 3.155	4.678	1.808
2.88	- 0.2646	- 3.1446	4.6847	1.8047
2.89	- 0.244	- 3.134	4.691	1.801
2.90	- 0.223	- 3.123	4.698	1.798
2.91	- 0.203	- 3.113	4.7046	1.7946
2.92	- 0.183	- 3.103	4.711	1.791
2.93	- 0.162	- 3.092	4.718	1.788
2.94	- 0.142	- 3.082	4.724	1.784
2.95	- 0.121	- 3.071	4.731	1.781
2.96	- 0.101	- 3.061	4.738	1.778
2.97	- 0.081	- 3.051	4.744	1.774
2.98	- 0.061	- 3.041	4.751	1.771
2.99	- 0.041	- 3.031	4.758	1.768
3.00	- 0.021	- 3.021	4.764	1.764
3.01	- 0.001	- 3.011	4.771	1.761
3.02	0.019	- 3.001	4.778	1.758
3.03	0.039	- 2.991	4.784	1.754
3.04	0.059	- 2.981	4.791	1.751
3.05	0.079	- 2.971	4.798	1.748
3.06	0.099	- 2.961	4.804	1.744
3.07	0.119	- 2.951	4.811	1.741
3.08	0.138	- 2.942	4.818	1.738
3.09	0.158	- 2.932	4.8245	1.7345
3.10	0.178	- 2.922	4.831	1.731
3.11	0.197	- 2.913	4.838	1.728
3.12	0.217	- 2.903	4.8446	1.7246
3.13	0.236	- 2.894	4.851	1.721
3.14	0.256	- 2.884	4.858	1.718
3.15	0.275	- 2.875	4.8648	1.7148
3.16	0.294	- 2.866	4.872	1.712
3.17	0.314	- 2.856	4.878	1.708
3.18	0.333	- 2.847	4.885	1.705
3.19	0.352	- 2.838	4.892	1.702
3.20	0.371	- 2.829	4.899	1.699

Table 10. Tables of L(u)(continued).

u	L(u) + Ɨ	L(-u)	L(u+Ɨ)	L(-u+Ɨ)
3.21	0.391	- 2.819	4.905	1.695
3.22	0.410	- 2.810	4.912	1.692
3.23	0.429	- 2.801	4.919	1.689
3.24	0.448	- 2.792	4.926	1.686
3.25	0.467	- 2.783	4.932	1.682
3.26	0.486	- 2.774	4.939	1.679
3.27	0.505	- 2.765	4.946	1.676
3.28	0.524	- 2.756	4.953	1.673
3.29	0.542	- 2.748	4.960	1.670
3.30	0.561	- 2.739	4.966	1.666
3.31	0.580	- 2.730	4.973	1.663
3.32	0.599	- 2.721	4.980	1.660
3.33	0.617	- 2.713	4.987	1.657
3.34	0.636	- 2.704	4.994	1.654
3.35	0.6547	- 2.695	5.001	1.651
3.36	0.673	- 2.687	5.007	1.647
3.37	0.692	- 2.678	5.014	1.644
3.38	0.710	- 2.670	5.021	1.641
3.39	0.729	- 2.661	5.028	1.638
3.40	0.747	- 2.653	5.0348	1.6348
3.41	0.766	- 2.644	5.042	1.632
3.42	0.784	- 2.636	5.049	1.629
3.43	0.802	- 2.628	5.055	1.625
3.44	0.821	- 2.619	5.062	1.622
3.45	0.839	- 2.611	5.069	1.619
3.46	0.857	- 2.603	5.076	1.616
3.47	0.875	- 2.5946	5.083	1.613
3.48	0.894	- 2.586	5.090	1.610
3.49	0.912	- 2.578	5.097	1.607
3.50	0.930	- 2.570	5.104	1.604
3.51	0.948	- 2.562	5.111	1.601
3.52	0.966	- 2.554	5.118	1.598
3.53	0.984	- 2.546	5.124	1.594
3.54	1.002	- 2.538	5.131	1.591
3.55	1.020	- 2.530	5.138	1.588
3.56	1.038	- 2.522	5.145	1.585
3.57	1.055	- 2.515	5.152	1.582
3.58	1.073	- 2.507	5.159	1.579
3.59	1.091	- 2.499	5.166	1.576
3.60	1.109	- 2.491	5.173	1.573

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
3.61	1.127	- 2.483	5.180	1.570
3.62	1.144	- 2.476	5.187	1.567
3.63	1.162	- 2.468	5.194	1.564
3.64	1.180	- 2.460	5.201	1.561
3.65	1.197	- 2.453	5.208	1.558
3.66	1.2147	- 2.445	5.2149	1.5549
3.67	1.232	- 2.438	5.222	1.552
3.68	1.250	- 2.430	5.229	1.549
3.69	1.267	- 2.423	5.236	1.546
3.70	1.2847	- 2.415	5.243	1.543
3.71	1.302	- 2.408	5.250	1.540
3.72	1.320	- 2.400	5.257	1.537
3.73	1.337	- 2.393	5.264	1.534
3.74	1.354	- 2.386	5.271	1.531
3.75	1.372	- 2.378	5.278	1.528
3.76	1.389	- 2.371	5.285	1.525
3.77	1.406	- 2.364	5.292	1.522
3.78	1.423	- 2.357	5.299	1.519
3.79	1.440	- 2.350	5.306	1.516
3.80	1.458	- 2.342	5.313	1.513
3.81	1.4747	- 2.335	5.320	1.510
3.82	1.492	- 2.328	5.327	1.507
3.83	1.509	- 2.321	5.334	1.504
3.84	1.526	- 2.314	5.342	1.502
3.85	1.543	- 2.307	5.349	1.499
3.86	1.560	- 2.300	5.356	1.496
3.87	1.577	- 2.293	5.363	1.493
3.88	1.594	- 2.286	5.370	1.490
3.89	1.611	- 2.279	5.377	1.487
3.90	1.628	- 2.272	5.384	1.484
3.91	1.6446	- 2.265	5.391	1.481
3.92	1.661	- 2.259	5.398	1.478
3.93	1.678	- 2.252	5.405	1.475
3.94	1.695	- 2.2449	5.413	1.473
3.95	1.712	- 2.238	5.420	1.470
3.96	1.729	- 2.231	5.427	1.467
3.97	1.745	- 2.2247	5.434	1.464
3.98	1.762	- 2.218	5.441	1.461
3.99	1.779	- 2.211	5.448	1.458
4.00	1.795	- 2.2048	5.455	1.455

Table 10. Tables of L(u)(continued).

u	L(u) + Ɨ	L(-u)	L(u+Ɨ)	L(-u+Ɨ)
4.01	1.812	- 2.198	5.463	1.453
4.02	1.828	- 2.192	5.470	1.450
4.03	1.8449	- 2.185	5.477	1.447
4.04	1.861	- 2.179	5.484	1.444
4.05	1.878	- 2.172	5.491	1.441
4.06	1.894	- 2.166	5.498	1.438
4.07	1.911	- 2.159	5.506	1.436
4.08	1.927	- 2.153	5.513	1.433
4.09	1.944	- 2.146	5.520	1.430
4.10	1.960	- 2.140	5.527	1.427
4.11	1.976	- 2.134	5.534	1.424
4.12	1.993	- 2.127	5.542	1.422
4.13	2.009	- 2.121	5.549	1.419
4.14	2.025	- 2.1146	5.556	1.416
4.15	2.042	- 2.108	5.563	1.413
4.16	2.058	- 2.102	5.570	1.410
4.17	2.074	- 2.096	5.578	1.408
4.18	2.090	- 2.090	5.5849	1.4049
4.19	2.106	- 2.084	5.592	1.402
4.20	2.123	- 2.077	5.599	1.399
4.21	2.139	- 2.071	5.607	1.397
4.22	2.1548	- 2.065	5.614	1.394
4.23	2.171	- 2.059	5.621	1.391
4.24	2.187	- 2.053	5.628	1.388
4.25	2.203	- 2.047	5.636	1.386
4.26	2.219	- 2.041	5.643	1.383
4.27	2.2349	- 2.035	5.650	1.380
4.28	2.251	- 2.029	5.657	1.377
4.29	2.267	- 2.023	5.6647	1.3747
4.30	2.283	- 2.017	5.672	1.372
4.31	2.299	- 2.011	5.679	1.369
4.32	2.3145	- 2.005	5.687	1.367
4.33	2.330	- 2.000	5.694	1.364
4.34	2.346	- 1.994	5.701	1.361
4.35	2.362	- 1.988	5.709	1.359
4.36	2.378	- 1.982	5.716	1.356
4.37	2.394	- 1.976	5.723	1.353
4.38	2.409	- 1.971	5.731	1.351
4.39	2.425	- 1.9648	5.738	1.348
4.40	2.441	- 1.959	5.745	1.345

Table 10. Tables of L(u)(continued).

u	L(u) + Ɨ	L(-u)	L(u+Ɨ)	L(-u+Ɨ)
4.41	2.457	- 1.953	5.753	1.343
4.42	2.472	- 1.948	5.760	1.340
4.43	2.488	- 1.942	5.767	1.337
4.44	2.504	- 1.936	5.7745	1.3345
4.45	2.519	- 1.931	5.782	1.332
4.46	2.5346	- 1.925	5.789	1.329
4.47	2.550	- 1.920	5.797	1.327
4.48	2.566	- 1.914	5.804	1.324
4.49	2.581	- 1.909	5.811	1.321
4.50	2.597	- 1.903	5.819	1.319
4.51	2.612	- 1.898	5.826	1.316
4.52	2.628	- 1.892	5.834	1.314
4.53	2.643	- 1.887	5.841	1.311
4.54	2.659	- 1.881	5.848	1.308
4.55	2.674	- 1.876	5.856	1.306
4.56	2.689	- 1.871	5.863	1.303
4.57	2.7048	- 1.865	5.871	1.301
4.58	2.720	- 1.860	5.878	1.298
4.59	2.736	- 1.854	5.885	1.295
4.60	2.751	- 1.849	5.893	1.293
4.61	2.766	- 1.844	5.900	1.290
4.62	2.781	- 1.839	5.908	1.288
4.63	2.797	- 1.833	5.915	1.285
4.64	2.812	- 1.828	5.923	1.283
4.65	2.827	- 1.823	5.930	1.280
4.66	2.842	- 1.818	5.937	1.277
4.67	2.858	- 1.812	5.9449	1.2749
4.68	2.873	- 1.807	5.952	1.272
4.69	2.888	- 1.802	5.960	1.270
4.70	2.903	- 1.797	5.967	1.267
4.71	2.918	- 1.792	5.9747	1.2647
4.72	2.933	- 1.787	5.982	1.262
4.73	2.948	- 1.782	5.990	1.260
4.74	2.963	- 1.777	5.997	1.257
4.75	2.978	- 1.772	6.0047	1.2547
4.76	2.993	- 1.767	6.012	1.252
4.77	3.008	- 1.762	6.020	1.250
4.78	3.023	- 1.757	6.027	1.247
4.79	3.038	- 1.752	6.0346	1.2446
4.80	3.053	- 1.747	6.042	1.242

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
4.81	3.068	- 1.742	6.050	1.240
4.82	3.083	- 1.737	6.057	1.237
4.83	3.098	- 1.732	6.0647	1.2347
4.84	3.113	- 1.727	6.072	1.232
4.85	3.128	- 1.722	6.080	1.230
4.86	3.143	- 1.717	6.087	1.227
4.87	3.158	- 1.712	6.0949	1.2249
4.88	3.172	- 1.708	6.102	1.222
4.89	3.187	- 1.703	6.110	1.220
4.90	3.202	- 1.698	6.118	1.218
4.91	3.217	- 1.693	6.125	1.215
4.92	3.232	- 1.688	6.133	1.213
4.93	3.246	- 1.684	6.140	1.210
4.94	3.261	- 1.679	6.148	1.208
4.95	3.276	- 1.674	6.155	1.205
4.96	3.290	- 1.670	6.163	1.203
4.97	3.305	- 1.6648	6.171	1.201
4.98	3.320	- 1.660	6.178	1.198
4.99	3.334	- 1.656	6.186	1.196
5.00	3.349	- 1.651	6.193	1.193
5.01	3.364	- 1.646	6.201	1.191
5.02	3.378	- 1.642	6.209	1.189
5.03	3.393	- 1.637	6.216	1.186
5.04	3.407	- 1.633	6.224	1.184
5.05	3.422	- 1.628	6.231	1.181
5.06	3.437	- 1.623	6.239	1.179
5.07	3.451	- 1.619	6.247	1.177
5.08	3.466	- 1.614	6.254	1.174
5.09	3.480	- 1.610	6.262	1.172
5.10	3.4945	- 1.605	6.269	1.169
5.11	3.509	- 1.601	6.277	1.167
5.12	3.524	- 1.596	6.2847	1.1647
5.13	3.538	- 1.592	6.292	1.162
5.14	3.552	- 1.588	6.300	1.160
5.15	3.567	- 1.583	6.308	1.158
5.16	3.581	- 1.579	6.315	1.155
5.17	3.596	- 1.574	6.323	1.153
5.18	3.610	- 1.570	6.331	1.151
5.19	3.624	- 1.566	6.338	1.148
5.20	3.639	- 1.561	6.346	1.146

Table 10. Tables of L(u)(continued).

u	L(u) + Ī	L(-u)	L(u+Ī)	L(-u+Ī)
5.21	3.653	- 1.557	6.354	1.144
5.22	3.667	- 1.553	6.361	1.141
5.23	3.681	- 1.549	6.369	1.139
5.24	3.696	- 1.544	6.377	1.137
5.25	3.710	- 1.540	6.3845	1.1345
5.26	3.724	- 1.536	6.392	1.132
5.27	3.739	- 1.531	6.400	1.130
5.28	3.753	- 1.527	6.408	1.128
5.29	3.767	- 1.523	6.415	1.125
5.30	3.781	- 1.519	6.423	1.123
5.31	3.795	- 1.5146	6.431	1.121
5.32	3.809	- 1.511	6.439	1.119
5.33	3.824	- 1.506	6.446	1.116
5.34	3.838	- 1.502	6.454	1.114
5.35	3.852	- 1.498	6.462	1.112
5.36	3.866	- 1.494	6.470	1.110
5.37	3.880	- 1.490	6.477	1.107
5.38	3.894	- 1.486	6.485	1.105
5.39	3.908	- 1.482	6.493	1.103
5.40	3.922	- 1.478	6.501	1.101
5.41	3.936	- 1.474	6.508	1.098
5.42	3.950	- 1.470	6.516	1.096
5.43	3.964	- 1.466	6.524	1.094
5.44	3.978	- 1.462	6.532	1.092
5.45	3.992	- 1.458	6.539	1.089
5.46	4.006	- 1.454	6.547	1.087
5.47	4.020	- 1.450	6.5549	1.0849
5.48	4.034	- 1.446	6.563	1.083
5.49	4.048	- 1.442	6.571	1.081
5.50	4.062	- 1.438	6.578	1.078
5.51	4.076	- 1.434	6.586	1.076
5.52	4.090	- 1.430	6.594	1.074
5.53	4.104	- 1.426	6.602	1.072
5.54	4.118	- 1.422	6.610	1.070
5.55	4.132	- 1.418	6.617	1.067
5.56	4.146	- 1.414	6.625	1.065
5.57	4.159	- 1.411	6.633	1.063
5.58	4.173	- 1.407	6.641	1.061
5.59	4.187	- 1.403	6.649	1.059
5.60	4.201	- 1.399	6.657	1.057

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
5.61	4.2146	- 1.395	6.664	1.054
5.62	4.228	- 1.392	6.672	1.052
5.63	4.242	- 1.388	6.680	1.050
5.64	4.256	- 1.384	6.688	1.048
5.65	4.270	- 1.380	6.696	1.046
5.66	4.283	- 1.377	6.704	1.044
5.67	4.297	- 1.373	6.712	1.042
5.68	4.311	- 1.369	6.719	1.039
5.69	4.3245	- 1.365	6.727	1.037
5.70	4.338	- 1.362	6.735	1.035
5.71	4.352	- 1.358	6.743	1.033
5.72	4.366	- 1.354	6.751	1.031
5.73	4.379	- 1.351	6.759	1.029
5.74	4.393	- 1.347	6.767	1.027
5.75	4.407	- 1.343	6.7745	1.0245
5.76	4.420	- 1.340	6.782	1.022
5.77	4.434	- 1.336	6.790	1.020
5.78	4.447	- 1.333	6.798	1.018
5.79	4.461	- 1.329	6.806	1.016
5.80	4.4745	- 1.325	6.814	1.014
5.81	4.488	- 1.322	6.822	1.012
5.82	4.502	- 1.318	6.830	1.010
5.83	4.515	- 1.3148	6.838	1.008
5.84	4.529	- 1.311	6.846	1.006
5.85	4.542	- 1.308	6.854	1.004
5.86	4.556	- 1.304	6.862	1.002
5.87	4.569	- 1.301	6.870	1.000
5.88	4.583	- 1.297	6.878	0.998
5.89	4.596	- 1.294	6.886	0.996
5.90	4.610	- 1.290	6.893	0.993
5.91	4.623	- 1.287	6.901	0.991
5.92	4.637	- 1.283	6.909	0.989
5.93	4.650	- 1.280	6.917	0.987
5.94	4.663	- 1.277	6.925	0.985
5.95	4.677	- 1.273	6.933	0.983
5.96	4.690	- 1.270	6.941	0.981
5.97	4.704	- 1.266	6.949	0.979
5.98	4.717	- 1.263	6.957	0.977
5.99	4.730	- 1.260	6.965	0.975
6.00	4.744	- 1.256	6.973	0.973

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
6.01	4.757	- 1.253	6.981	0.971
6.02	4.770	- 1.250	6.989	0.969
6.03	4.784	- 1.246	6.997	0.967
6.04	4.797	- 1.243	7.005	0.965
6.05	4.810	- 1.240	7.013	0.963
6.06	4.824	- 1.236	7.021	0.961
6.07	4.837	- 1.233	7.029	0.959
6.08	4.850	- 1.230	7.037	0.957
6.09	4.864	- 1.226	7.045	0.955
6.10	4.877	- 1.223	7.053	0.953
6.11	4.890	- 1.220	7.061	0.951
6.12	4.903	- 1.217	7.069	0.949
6.13	4.916	- 1.214	7.077	0.947
6.14	4.930	- 1.210	7.085	0.945
6.15	4.943	- 1.207	7.094	0.944
6.16	4.956	- 1.204	7.102	0.942
6.17	4.969	- 1.201	7.110	0.940
6.18	4.982	- 1.198	7.118	0.938
6.19	4.996	- 1.194	7.126	0.936
6.20	5.009	- 1.191	7.134	0.934
6.21	5.022	- 1.188	7.142	0.932
6.22	5.035	- 1.1849	7.150	0.930
6.23	5.048	- 1.182	7.158	0.928
6.24	5.061	- 1.179	7.166	0.926
6.25	5.074	- 1.176	7.174	0.924
6.26	5.088	- 1.172	7.182	0.922
6.27	5.101	- 1.169	7.190	0.920
6.28	5.114	- 1.166	7.198	0.918
6.29	5.127	- 1.163	7.207	0.917
6.30	5.140	- 1.160	7.2146	0.9146
6.31	5.153	- 1.157	7.223	0.913
6.32	5.166	- 1.154	7.231	0.911
6.33	5.179	- 1.151	7.239	0.909
6.34	5.192	- 1.148	7.247	0.907
6.35	5.205	- 1.1449	7.255	0.905
6.36	5.218	- 1.142	7.263	0.903
6.37	5.231	- 1.139	7.271	0.901
6.38	5.244	- 1.136	7.280	0.900
6.39	5.257	- 1.133	7.288	0.898
6.40	5.270	- 1.130	7.296	0.896

Table 10. Tables of L(u)(continued).

u	L(u) +I	L(-u)	L(u+I)	L(-u+I)
6.41	5.283	- 1.127	7.304	0.894
6.42	5.296	- 1.124	7.312	0.892
6.43	5.309	- 1.121	7.320	0.890
6.44	5.322	- 1.118	7.328	0.888
6.45	5.3348	- 1.115	7.337	0.887
6.46	5.348	- 1.112	7.3447	0.8847
6.47	5.361	- 1.109	7.353	0.883
6.48	5.374	- 1.106	7.361	0.881
6.49	5.386	- 1.104	7.369	0.879
6.50	5.399	- 1.101	7.377	0.877
6.51	5.412	- 1.098	7.386	0.876
6.52	5.425	- 1.0949	7.394	0.874
6.53	5.438	- 1.092	7.402	0.872
6.54	5.451	- 1.089	7.410	0.870
6.55	5.464	- 1.086	7.418	0.868
6.56	5.476	- 1.084	7.426	0.866
6.57	5.489	- 1.081	7.4346	0.8646
6.58	5.502	- 1.078	7.443	0.863
6.59	5.5149	- 1.075	7.451	0.861
6.60	5.528	- 1.072	7.459	0.859
6.61	5.541	- 1.069	7.467	0.857
6.62	5.553	- 1.067	7.476	0.856
6.63	5.566	- 1.064	7.484	0.854
6.64	5.579	- 1.061	7.492	0.852
6.65	5.592	- 1.058	7.500	0.850
6.66	5.604	- 1.056	7.509	0.849
6.67	5.614	- 1.053	7.517	0.847
6.68	5.630	- 1.050	7.5249	0.8449
6.69	5.643	- 1.047	7.533	0.843
6.70	5.655	- 1.0446	7.541	0.841
6.71	5.668	- 1.042	7.550	0.840
6.72	5.681	- 1.039	7.558	0.838
6.73	5.693	- 1.037	7.566	0.836
6.74	5.706	- 1.034	7.574	0.834
6.75	5.719	- 1.031	7.583	0.833
6.76	5.732	- 1.028	7.591	0.831
6.77	5.744	- 1.026	7.599	0.829
6.78	5.757	- 1.023	7.607	0.827
6.79	5.770	- 1.020	7.616	0.826
6.80	5.782	- 1.018	7.624	0.824

Table 10. Tables of L(u)(continued).

u	L(u) + Ɨ	L(-u)	L(u+Ɨ)	L(-u+Ɨ)
6.81	5.7947	- 1.015	7.632	0.822
6.82	5.807	- 1.013	7.641	0.821
6.83	5.820	- 1.010	7.649	0.819
6.84	5.833	- 1.007	7.657	0.817
6.85	5.845	- 1.0047	7.665	0.815
6.86	5.858	- 1.002	7.674	0.814
6.87	5.870	- 1.000	7.682	0.812
6.88	5.883	- 0.997	7.690	0.810
6.89	5.896	- 0.994	7.699	0.809
6.90	5.908	- 0.992	7.707	0.807
6.91	5.921	- 0.989	7.715	0.805
6.92	5.933	- 0.987	7.724	0.804
6.93	5.946	- 0.984	7.732	0.802
6.94	5.958	- 0.982	7.740	0.800
6.95	5.971	- 0.979	7.749	0.799
6.96	5.983	- 0.977	7.757	0.797
6.97	5.996	- 0.974	7.765	0.795
6.98	6.008	- 0.972	7.773	0.793
6.99	6.021	- 0.969	7.782	0.792
7.00	6.033	- 0.967	7.790	0.790
7.01	6.046	- 0.964	7.798	0.788
7.02	6.058	- 0.962	7.807	0.787
7.03	6.071	- 0.959	7.815	0.785
7.04	6.083	- 0.957	7.823	0.783
7.05	6.096	- 0.954	7.832	0.782
7.06	6.108	- 0.952	7.840	0.780
7.07	6.121	- 0.949	7.849	0.779
7.08	6.133	- 0.947	7.857	0.777
7.09	6.146	- 0.944	7.865	0.775
7.10	6.158	- 0.942	7.874	0.774
7.11	6.170	- 0.940	7.882	0.772
7.12	6.183	- 0.937	7.890	0.770
7.13	6.195	- 0.9347	7.899	0.769
7.14	6.208	- 0.932	7.907	0.767
7.15	6.220	- 0.930	7.916	0.766
7.16	6.232	- 0.928	7.924	0.764
7.17	6.2448	- 0.925	7.933	0.763
7.18	6.257	- 0.923	7.941	0.761
7.19	6.270	- 0.920	7.949	0.759
7.20	6.282	- 0.918	7.957	0.757

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
7.21	6.294	- 0.916	7.966	0.756
7.22	6.307	- 0.913	7.974	0.754
7.23	6.319	- 0.911	7.983	0.753
7.24	6.331	- 0.909	7.991	0.751
7.25	6.344	- 0.906	7.999	0.749
7.26	6.356	- 0.904	8.008	0.748
7.27	6.368	- 0.902	8.016	0.746
7.28	6.381	- 0.899	8.0247	0.7447
7.29	6.393	- 0.897	8.033	0.743
7.30	6.405	- 0.8948	8.042	0.742
7.31	6.417	- 0.893	8.050	0.740
7.32	6.430	- 0.890	8.058	0.738
7.33	6.442	- 0.888	8.067	0.737
7.34	6.454	- 0.886	8.075	0.735
7.35	6.466	- 0.884	8.084	0.734
7.36	6.479	- 0.881	8.092	0.732
7.37	6.491	- 0.879	8.101	0.731
7.38	6.503	- 0.877	8.109	0.729
7.39	6.515	- 0.8745	8.118	0.728
7.40	6.528	- 0.872	8.126	0.726
7.41	6.540	- 0.870	8.1345	0.7245
7.42	6.552	- 0.868	8.143	0.723
7.43	6.564	- 0.866	8.151	0.721
7.44	6.577	- 0.863	8.160	0.720
7.45	6.589	- 0.861	8.168	0.718
7.46	6.601	- 0.859	8.177	0.717
7.47	6.613	- 0.857	8.185	0.715
7.48	6.625	- 0.8547	8.914	0.714
7.49	6.637	- 0.853	8.202	0.712
7.50	6.650	- 0.850	8.211	0.711
7.51	6.662	- 0.848	8.219	0.709
7.52	6.674	- 0.846	8.228	0.708
7.53	6.686	- 0.844	8.236	0.706
7.54	6.698	- 0.842	8.2448	0.7048
7.55	6.710	- 0.840	8.253	0.703
7.56	6.722	- 0.838	8.262	0.702
7.57	6.7346	- 0.835	8.270	0.700
7.58	6.747	- 0.833	8.279	0.699
7.59	6.759	- 0.831	8.287	0.697
7.60	6.771	- 0.829	8.296	0.696

Table 10. Tables of L(u)(continued).

u	L(u) + ½	L(-u)	L(u+½)	L(-u+½)
7.61	6.783	- 0.827	8.304	0.694
7.62	6.795	- 0.8248	8.313	0.693
7.63	6.807	- 0.823	8.321	0.691
7.64	6.819	- 0.821	8.330	0.690
7.65	6.831	- 0.819	8.339	0.689
7.66	6.843	- 0.817	8.347	0.687
7.67	6.856	- 0.814	8.356	0.686
7.68	6.868	- 0.812	8.364	0.684
7.69	6.880	- 0.810	8.373	0.683
7.70	6.892	- 0.808	8.381	0.681
7.71	6.904	- 0.806	8.390	0.680
7.72	6.916	- 0.804	8.398	0.678
7.73	6.928	- 0.802	8.407	0.677
7.74	6.940	- 0.800	8.415	0.675
7.75	6.952	- 0.798	8.424	0.674
7.76	6.964	- 0.796	8.433	0.673
7.77	6.976	- 0.794	8.441	0.671
7.78	6.988	- 0.792	8.450	0.670
7.79	7.000	- 0.790	8.458	0.668
7.80	7.012	- 0.788	8.467	0.667
7.81	7.024	- 0.786	8.475	0.665
7.82	7.036	- 0.784	8.484	0.664
7.83	7.048	- 0.782	8.493	0.663
7.84	7.060	- 0.780	8.501	0.661
7.85	7.072	- 0.778	8.510	0.660
7.86	7.084	- 0.776	8.518	0.658
7.87	7.096	- 0.774	8.527	0.657
7.88	7.108	- 0.772	8.536	0.656
7.89	7.120	- 0.770	8.544	0.654
7.90	7.132	- 0.768	8.553	0.653
7.91	7.143	- 0.767	8.561	0.651
7.92	7.155	- 0.7646	8.570	0.650
7.93	7.167	- 0.763	8.579	0.649
7.94	7.179	- 0.761	8.587	0.647
7.95	7.191	- 0.759	8.596	0.646
7.96	7.203	- 0.757	8.604	0.644
7.97	7.2149	- 0.755	8.613	0.643
7.98	7.227	- 0.753	8.622	0.642
7.99	7.239	- 0.751	8.630	0.640
8.00	7.251	- 0.749	8.639	0.639

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
8.01	7.262	- 0.748	8.648	0.638
8.02	7.274	- 0.746	8.656	0.636
8.03	7.286	- 0.744	8.6648	0.6348
8.04	7.298	- 0.742	8.673	0.633
8.05	7.310	- 0.740	8.682	0.632
8.06	7.322	- 0.738	8.691	0.631
8.07	7.334	- 0.736	8.699	0.629
8.08	7.345	- 0.7345	8.708	0.628
8.09	7.357	- 0.733	8.717	0.627
8.10	7.369	- 0.731	8.725	0.625
8.11	7.381	- 0.729	8.734	0.624
8.12	7.393	- 0.727	8.743	0.623
8.13	7.4046	- 0.725	8.751	0.621
8.14	7.416	- 0.724	8.760	0.620
8.15	7.428	- 0.722	8.769	0.619
8.16	7.440	- 0.720	8.777	0.617
8.17	7.452	- 0.718	8.786	0.616
8.18	7.464	- 0.716	8.7947	0.6147
8.19	7.475	- 0.7145	8.803	0.613
8.20	7.487	- 0.713	8.812	0.612
8.21	7.499	- 0.711	8.821	0.611
8.22	7.511	- 0.709	8.829	0.609
8.23	7.523	- 0.707	8.838	0.608
8.24	7.534	- 0.706	8.847	0.607
8.25	7.546	- 0.704	8.856	0.606
8.26	7.558	- 0.702	8.864	0.604
8.27	7.570	- 0.700	8.873	0.603
8.28	7.581	- 0.699	8.882	0.602
8.29	7.593	- 0.697	8.890	0.600
8.30	7.6048	- 0.695	8.899	0.599
8.31	7.617	- 0.693	8.908	0.598
8.32	7.628	- 0.692	8.917	0.597
8.33	7.640	- 0.690	8.925	0.595
8.34	7.652	- 0.688	8.934	0.594
8.35	7.663	- 0.687	8.943	0.593
8.36	7.675	- 0.6848	8.951	0.591
8.37	7.687	- 0.683	8.960	0.590
8.38	7.699	- 0.681	8.969	0.589
8.39	7.710	- 0.680	8.978	0.588
8.40	7.722	- 0.678	8.986	0.586

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
8.41	7.734	- 0.676	8.995	0.585
8.42	7.745	- 0.6746	9.004	0.584
8.43	7.757	- 0.673	9.013	0.583
8.44	7.769	- 0.671	9.021	0.581
8.45	7.780	- 0.670	9.030	0.580
8.46	7.792	- 0.668	9.039	0.579
8.47	7.804	- 0.666	9.048	0.578
8.48	7.815	- 0.6646	9.056	0.576
8.49	7.827	- 0.663	9.065	0.575
8.50	7.839	- 0.661	9.074	0.574
8.51	7.850	- 0.660	9.083	0.573
8.52	7.862	- 0.658	9.091	0.571
8.53	7.874	- 0.656	9.100	0.570
8.54	7.885	- 0.6548	9.109	0.569
8.55	7.897	- 0.653	9.118	0.568
8.56	7.908	- 0.652	9.126	0.566
8.57	7.920	- 0.650	9.135	0.565
8.58	7.932	- 0.648	9.144	0.564
8.59	7.943	- 0.647	9.153	0.563
8.60	7.9548	- 0.645	9.162	0.562
8.61	7.966	- 0.644	9.170	0.560
8.62	7.978	- 0.642	9.179	0.559
8.63	7.990	- 0.640	9.188	0.558
8.64	8.001	- 0.639	9.197	0.557
8.65	8.013	- 0.637	9.206	0.556
8.66	8.024	- 0.636	9.214	0.554
8.67	8.036	- 0.634	9.223	0.553
8.68	8.048	- 0.632	9.232	0.552
8.69	8.059	- 0.631	9.241	0.551
8.70	8.071	- 0.629	9.250	0.550
8.71	8.082	- 0.628	9.258	0.548
8.72	8.094	- 0.626	9.267	0.547
8.73	8.105	- 0.6246	9.276	0.546
8.74	8.117	- 0.623	9.2848	0.5448
8.75	8.128	- 0.622	9.294	0.544
8.76	8.140	- 0.620	9.302	0.542
8.77	8.151	- 0.619	9.311	0.541
8.78	8.163	- 0.617	9.320	0.540
8.79	8.1745	- 0.615	9.329	0.539
8.80	8.186	- 0.614	9.338	0.538

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
8.81	8.198	- 0.612	9.347	0.537
8.82	8.209	- 0.611	9.355	0.535
8.83	8.221	- 0.609	9.364	0.534
8.84	8.232	- 0.608	9.373	0.533
8.85	8.244	- 0.606	9.382	0.532
8.86	8.255	- 0.6048	9.391	0.531
8.87	8.267	- 0.603	9.400	0.530
8.88	8.278	- 0.602	9.409	0.529
8.89	8.290	- 0.600	9.417	0.527
8.90	8.301	- 0.599	9.426	0.526
8.91	8.313	- 0.597	9.435	0.525
8.92	8.324	- 0.596	9.444	0.524
8.93	8.335	- 0.5945	9.453	0.523
8.94	8.347	- 0.593	9.462	0.522
8.95	8.358	- 0.592	9.471	0.521
8.96	8.370	- 0.590	9.479	0.519
8.97	8.381	- 0.589	9.488	0.518
8.98	8.393	- 0.587	9.497	0.517
8.99	8.404	- 0.586	9.506	0.516
9.00	8.416	- 0.584	9.5149	0.5149
9.01	8.427	- 0.583	9.524	0.514
9.02	8.439	- 0.581	9.533	0.513
9.03	8.450	- 0.580	9.542	0.512
9.04	8.461	- 0.579	9.551	0.511
9.05	8.473	- 0.577	9.559	0.509
9.06	8.484	- 0.576	9.568	0.508
9.07	8.496	- 0.574	9.577	0.507
9.08	8.507	- 0.573	9.586	0.506
9.09	8.518	- 0.572	9.5949	0.5049
9.10	8.530	- 0.570	9.604	0.504
9.11	8.541	- 0.569	9.613	0.503
9.12	8.553	- 0.567	9.622	0.502
9.13	8.564	- 0.566	9.631	0.501
9.14	8.575	- 0.5645	9.640	0.500
9.15	8.587	- 0.563	9.648	0.498
9.16	8.598	- 0.562	9.657	0.497
9.17	8.610	- 0.560	9.666	0.496
9.18	8.621	- 0.559	9.675	0.495
9.19	8.632	- 0.558	9.684	0.494
9.20	8.644	- 0.556	9.693	0.493

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
9.21	8.655	- 0.5549	9.702	0.492
9.22	8.666	- 0.554	9.711	0.491
9.23	8.678	- 0.552	9.720	0.490
9.24	8.689	- 0.551	9.729	0.489
9.25	8.700	- 0.550	9.738	0.488
9.26	8.712	- 0.548	9.747	0.487
9.27	8.723	- 0.547	9.756	0.486
9.28	8.7345	- 0.545	9.7645	0.4845
9.29	8.746	- 0.544	9.773	0.483
9.30	8.757	- 0.543	9.782	0.482
9.31	8.769	- 0.541	9.791	0.481
9.32	8.780	- 0.540	9.800	0.480
9.33	8.791	- 0.539	9.809	0.479
9.34	8.802	- 0.538	9.818	0.478
9.35	8.814	- 0.536	9.827	0.477
9.36	8.825	- 0.5348	9.836	0.476
9.37	8.836	- 0.534	9.845	0.475
9.38	8.848	- 0.532	9.854	0.474
9.39	8.859	- 0.531	9.863	0.473
9.40	8.870	- 0.530	9.872	0.472
9.41	8.882	- 0.528	9.881	0.471
9.42	8.893	- 0.527	9.890	0.470
9.43	8.904	- 0.526	9.899	0.469
9.44	8.916	- 0.524	9.908	0.468
9.45	8.927	- 0.523	9.917	0.467
9.46	8.938	- 0.522	9.926	0.466
9.47	8.949	- 0.521	9.9348	0.4648
9.48	8.961	- 0.519	9.944	0.464
9.49	8.972	- 0.518	9.953	0.463
9.50	8.983	- 0.517	9.962	0.462
9.51	8.994	- 0.516	9.971	0.461
9.52	9.006	- 0.514	9.980	0.460
9.53	9.017	- 0.513	9.989	0.459
9.54	9.028	- 0.512	9.998	0.458
9.55	9.039	- 0.511	10.007	0.457
9.56	9.051	- 0.509	10.016	0.456
9.57	9.062	- 0.508	10.0248	0.4548
9.58	9.073	- 0.507	10.034	0.454
9.59	9.084	- 0.506	10.043	0.453
9.60	9.096	- 0.504	10.052	0.452

Table 10. Tables of L(u)(continued).

u	L(u) + Ɨ	L(-u)	L(u+Ɨ)	L(-u+Ɨ)
9.61	9.107	- 0.503	10.061	0.451
9.62	9.118	- 0.502	10.070	0.450
9.63	9.129	- 0.501	10.079	0.449
9.64	9.141	- 0.499	10.088	0.448
9.65	9.152	- 0.498	10.097	0.447
9.66	9.163	- 0.497	10.106	0.446
9.67	9.174	- 0.496	10.1149	0.4449
9.68	9.185	- 0.4946	10.124	0.444
9.69	9.197	- 0.493	10.133	0.443
9.70	9.208	- 0.492	10.142	0.442
9.71	9.219	- 0.491	10.151	0.441
9.72	9.230	- 0.490	10.160	0.440
9.73	9.241	- 0.489	10.169	0.439
9.74	9.253	- 0.487	10.178	0.438
9.75	9.264	- 0.486	10.187	0.437
9.76	9.2749	- 0.485	10.196	0.436
9.77	9.286	- 0.484	10.205	0.435
9.78	9.297	- 0.483	10.214	0.434
9.79	9.308	- 0.482	10.223	0.433
9.80	9.320	- 0.480	10.232	0.432
9.81	9.331	- 0.479	10.242	0.432
9.82	9.342	- 0.478	10.251	0.431
9.83	9.353	- 0.477	10.260	0.430
9.84	9.364	- 0.476	10.269	0.429
9.85	9.375	- 0.4745	10.278	0.428
9.86	9.387	- 0.473	10.287	0.427
9.87	9.398	- 0.472	10.296	0.426
9.88	9.409	- 0.471	10.3049	0.4249
9.89	9.420	- 0.470	10.314	0.424
9.90	9.431	- 0.469	10.323	0.423
9.91	9.442	- 0.468	10.332	0.422
9.92	9.453	- 0.467	10.341	0.421
9.93	9.4645	- 0.465	10.350	0.420
9.94	9.476	- 0.464	10.359	0.419
9.95	9.487	- 0.463	10.368	0.418
9.96	9.498	- 0.462	10.378	0.418
9.97	9.509	- 0.461	10.387	0.417
9.98	9.520	- 0.460	10.396	0.416
9.99	9.531	- 0.459	10.4048	0.4148
10.00	9.542	- 0.458	10.414	0.414

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
10.01	9.554	- 0.456	10.423	0.413
10.02	9.5646	- 0.455	10.432	0.412
10.03	9.576	- 0.454	10.441	0.411
10.04	9.587	- 0.453	10.450	0.410
10.05	9.598	- 0.452	10.459	0.409
10.06	9.609	- 0.451	10.469	0.409
10.07	9.620	- 0.450	10.478	0.408
10.08	9.631	- 0.449	10.487	0.407
10.09	9.642	- 0.448	10.496	0.406
10.10	9.653	- 0.447	10.5049	0.4049
10.11	9.664	- 0.446	10.514	0.404
10.12	9.676	- 0.444	10.523	0.403
10.13	9.687	- 0.443	10.532	0.402
10.14	9.698	- 0.442	10.541	0.401
10.15	9.709	- 0.441	10.551	0.401
10.16	9.720	- 0.440	10.560	0.400
10.17	9.731	- 0.439	10.569	0.399
10.18	9.742	- 0.438	10.578	0.398
10.19	9.753	- 0.437	10.587	0.397
10.20	9.764	- 0.436	10.596	0.396
10.21	9.775	- 0.4348	10.605	0.395
10.22	9.786	- 0.434	10.614	0.394
10.23	9.797	- 0.433	10.624	0.394
10.24	9.808	- 0.432	10.633	0.393
10.25	9.819	- 0.431	10.642	0.392
10.26	9.830	- 0.430	10.651	0.391
10.27	9.841	- 0.429	10.660	0.390
10.28	9.852	- 0.428	10.669	0.389
10.29	9.863	- 0.427	10.678	0.388
10.30	9.8745	- 0.425	10.687	0.387
10.31	9.886	- 0.424	10.697	0.387
10.32	9.897	- 0.423	10.706	0.386
10.33	9.908	- 0.422	10.7149	0.3849
10.34	9.919	- 0.421	10.724	0.384
10.35	9.930	- 0.420	10.733	0.383
10.36	9.941	- 0.419	10.742	0.382
10.37	9.952	- 0.418	10.752	0.382
10.38	9.963	- 0.417	10.761	0.381
10.39	9.974	- 0.416	10.770	0.380
10.40	9.9846	- 0.415	10.779	0.379

Table 10. Tables of L(u)(continued).

u	L(u) + Ī	L(-u)	L(u+Ī)	L(-u+Ī)
10.41	9.996	- 0.414	10.788	0.378
10.42	10.007	- 0.413	10.797	0.377
10.43	10.018	- 0.412	10.807	0.377
10.44	10.029	- 0.411	10.816	0.376
10.45	10.040	- 0.410	10.8248	0.3748
10.46	10.051	- 0.409	10.834	0.374
10.47	10.062	- 0.408	10.843	0.373
10.48	10.073	- 0.407	10.852	0.372
10.49	10.084	- 0.406	10.862	0.372
10.50	10.0945	- 0.405	10.871	0.371
10.51	10.106	- 0.404	10.880	0.370
10.52	10.117	- 0.403	10.889	0.369
10.53	10.128	- 0.402	10.898	0.368
10.54	10.138	- 0.402	10.908	0.368
10.55	10.149	- 0.401	10.917	0.367
10.56	10.160	- 0.400	10.926	0.366
10.57	10.171	- 0.399	10.935	0.365
10.58	10.182	- 0.398	10.944	0.364
10.59	10.193	- 0.397	10.953	0.363
10.60	10.204	- 0.396	10.963	0.363
10.61	10.215	- 0.3947	10.972	0.362
10.62	10.226	- 0.394	10.981	0.361
10.63	10.237	- 0.393	10.990	0.360
10.64	10.248	- 0.392	10.999	0.359
10.65	10.259	- 0.391	11.009	0.359
10.66	10.270	- 0.390	11.018	0.358
10.67	10.281	- 0.389	11.027	0.357
10.68	10.292	- 0.388	11.036	0.356
10.69	10.303	- 0.387	11.046	0.356
10.70	10.314	- 0.386	11.0547	0.3547
10.71	10.3246	- 0.385	11.064	0.354
10.72	10.336	- 0.384	11.073	0.353
10.73	10.346	- 0.384	11.082	0.352
10.74	10.357	- 0.383	11.092	0.352
10.75	10.368	- 0.382	11.101	0.351
10.76	10.379	- 0.381	11.110	0.350
10.77	10.390	- 0.380	11.119	0.349
10.78	10.401	- 0.379	11.129	0.349
10.79	10.412	- 0.378	11.138	0.348
10.80	10.423	- 0.377	11.147	0.347

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
10.81	10.434	- 0.376	11.156	0.346
10.82	10.4446	- 0.375	11.165	0.345
10.83	10.456	- 0.374	11.1746	0.3446
10.84	10.466	- 0.374	11.184	0.344
10.85	10.477	- 0.373	11.193	0.343
10.86	10.488	- 0.372	11.202	0.342
10.87	10.499	- 0.371	11.212	0.342
10.88	10.510	- 0.370	11.221	0.341
10.89	10.521	- 0.369	11.230	0.340
10.90	10.532	- 0.368	11.239	0.339
10.91	10.543	- 0.367	11.249	0.339
10.92	10.554	- 0.366	11.258	0.338
10.93	10.564	- 0.366	11.267	0.337
10.94	10.575	- 0.3646	11.276	0.336
10.95	10.586	- 0.364	11.286	0.336
10.96	10.597	- 0.363	11.2949	0.3349
10.97	10.608	- 0.362	11.304	0.334
10.98	10.619	- 0.361	11.313	0.333
10.99	10.630	- 0.360	11.323	0.333
11.00	10.641	- 0.359	11.332	0.332
11.01	10.651	- 0.359	11.341	0.331
11.02	10.662	- 0.358	11.350	0.330
11.03	10.673	- 0.357	11.360	0.330
11.04	10.684	- 0.356	11.369	0.329
11.05	10.6948	- 0.355	11.378	0.328
11.06	10.706	- 0.354	11.388	0.328
11.07	10.717	- 0.353	11.397	0.327
11.08	10.727	- 0.353	11.406	0.326
11.09	10.738	- 0.352	11.415	0.325
11.10	10.749	- 0.351	11.4246	0.3246
11.11	10.760	- 0.350	11.434	0.324
11.12	10.771	- 0.349	11.443	0.323
11.13	10.782	- 0.348	11.453	0.323
11.14	10.792	- 0.348	11.462	0.322
11.15	10.803	- 0.347	11.471	0.321
11.16	10.814	- 0.346	11.480	0.320
11.17	10.8249	- 0.345	11.490	0.320
11.18	10.836	- 0.344	11.499	0.319
11.19	10.847	- 0.343	11.508	0.318
11.20	10.857	- 0.343	11.518	0.318

Table 10. Tables of L(u)(continued).

u	L(u) + Ī	L(-u)	L(u+Ī)	L(-u+Ī)
11.21	10.868	- 0.342	11.527	0.317
11.22	10.879	- 0.341	11.536	0.316
11.23	10.890	- 0.340	11.545	0.315
11.24	10.901	- 0.339	11.5547	0.3147
11.25	10.911	- 0.339	11.564	0.314
11.26	10.922	- 0.338	11.573	0.313
11.27	10.933	- 0.337	11.583	0.313
11.28	10.944	- 0.336	11.592	0.312
11.29	10.9546	- 0.335	11.601	0.311
11.30	10.965	- 0.3345	11.611	0.311
11.31	10.976	- 0.334	11.620	0.310
11.32	10.987	- 0.333	11.629	0.309
11.33	10.998	- 0.332	11.639	0.309
11.34	11.009	- 0.331	11.648	0.308
11.35	11.019	- 0.331	11.657	0.307
11.36	11.030	- 0.330	11.666	0.306
11.37	11.041	- 0.329	11.676	0.306
11.38	11.052	- 0.328	11.685	0.305
11.39	11.063	- 0.327	11.694	0.304
11.40	11.073	- 0.327	11.704	0.304
11.41	11.084	- 0.326	11.713	0.303
11.42	11.0949	- 0.325	11.722	0.302
11.43	11.106	- 0.324	11.732	0.302
11.44	11.117	- 0.323	11.741	0.301
11.45	11.127	- 0.323	11.750	0.300
11.46	11.138	- 0.322	11.760	0.300
11.47	11.149	- 0.321	11.769	0.299
11.48	11.160	- 0.320	11.778	0.298
11.49	11.170	- 0.320	11.788	0.298
11.50	11.181	- 0.319	11.797	0.297
11.51	11.192	- 0.318	11.806	0.296
11.52	11.203	- 0.317	11.816	0.296
11.53	11.213	- 0.317	11.825	0.295
11.54	11.224	- 0.316	11.834	0.294
11.55	11.2349	- 0.315	11.844	0.294
11.56	11.246	- 0.314	11.853	0.293
11.57	11.256	- 0.314	11.862	0.292
11.58	11.267	- 0.313	11.872	0.292
11.59	11.278	- 0.312	11.881	0.291
11.60	11.289	- 0.311	11.891	0.291

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
11.61	11.299	- 0.311	11.900	0.290
11.62	11.310	- 0.310	11.909	0.289
11.63	11.321	- 0.309	11.919	0.289
11.64	11.332	- 0.308	11.928	0.288
11.65	11.342	- 0.308	11.937	0.287
11.66	11.353	- 0.307	11.947	0.287
11.67	11.364	- 0.306	11.956	0.286
11.68	11.3745	- 0.305	11.965	0.285
11.69	11.385	- 0.3047	11.9747	0.2847
11.70	11.396	- 0.304	11.984	0.284
11.71	11.407	- 0.303	11.993	0.283
11.72	11.417	- 0.303	12.003	0.283
11.73	11.428	- 0.302	12.012	0.282
11.74	11.439	- 0.301	12.022	0.282
11.75	11.450	- 0.300	12.031	0.281
11.76	11.460	- 0.300	12.040	0.280
11.77	11.471	- 0.299	12.050	0.280
11.78	11.482	- 0.298	12.059	0.279
11.79	11.492	- 0.298	12.068	0.278
11.80	11.503	- 0.297	12.078	0.278
11.81	11.514	- 0.296	12.087	0.277
11.82	11.5245	- 0.295	12.097	0.277
11.83	11.535	- 0.2947	12.106	0.276
11.84	11.546	- 0.294	12.115	0.275
11.85	11.557	- 0.293	12.1247	0.2747
11.86	11.567	- 0.293	12.134	0.274
11.87	11.578	- 0.292	12.144	0.274
11.88	11.589	- 0.291	12.153	0.273
11.89	11.599	- 0.291	12.162	0.272
11.90	11.610	- 0.290	12.172	0.272
11.91	11.621	- 0.289	12.181	0.271
11.92	11.632	- 0.288	12.191	0.271
11.93	11.642	- 0.288	12.200	0.270
11.94	11.653	- 0.287	12.209	0.269
11.95	11.664	- 0.286	12.219	0.269
11.96	11.674	- 0.286	12.228	0.268
11.97	11.6849	- 0.285	12.238	0.268
11.98	11.696	- 0.284	12.247	0.267
11.99	11.706	- 0.284	12.256	0.266
12.00	11.717	- 0.283	12.266	0.266

Table 10. Tables of L(u) (continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
12.01	11.728	- 0.282	12.275	0.265
12.02	11.738	- 0.282	12.2845	0.2645
12.03	11.749	- 0.281	12.294	0.264
12.04	11.760	- 0.280	12.303	0.263
12.05	11.770	- 0.280	12.313	0.263
12.06	11.781	- 0.279	12.322	0.262
12.07	11.792	- 0.278	12.332	0.262
12.08	11.802	- 0.278	12.341	0.261
12.09	11.813	- 0.277	12.350	0.260
12.10	11.824	- 0.276	12.360	0.260
12.11	11.834	- 0.276	12.369	0.259
12.12	11.8449	- 0.275	12.379	0.259
12.13	11.856	- 0.274	12.388	0.258
12.14	11.866	- 0.274	12.398	0.258
12.15	11.877	- 0.273	12.407	0.257
12.16	11.888	- 0.272	12.416	0.256
12.17	11.898	- 0.272	12.426	0.256
12.18	11.909	- 0.271	12.435	0.255
12.19	11.919	- 0.271	12.4446	0.2546
12.20	11.930	- 0.270	12.454	0.254
12.21	11.941	- 0.269	12.464	0.254
12.22	11.951	- 0.269	12.473	0.253
12.23	11.962	- 0.268	12.482	0.252
12.24	11.973	- 0.267	12.492	0.252
12.25	11.983	- 0.267	12.501	0.251
12.26	11.994	- 0.266	12.511	0.251
12.27	12.0045	- 0.265	12.520	0.250
12.28	12.015	- 0.2648	12.530	0.250
12.29	12.026	- 0.264	12.539	0.249
12.30	12.036	- 0.264	12.548	0.248
12.31	12.047	- 0.263	12.558	0.248
12.32	12.058	- 0.262	12.567	0.247
12.33	12.068	- 0.262	12.577	0.247
12.34	12.079	- 0.261	12.586	0.246
12.35	12.090	- 0.260	12.596	0.246
12.36	12.100	- 0.260	12.605	0.245
12.37	12.111	- 0.259	12.6146	0.2446
12.38	12.121	- 0.259	12.624	0.244
12.39	12.132	- 0.258	12.634	0.244
12.40	12.143	- 0.257	12.643	0.243

Table 10. Tables of L(u)(continued).

u	L(u) + Ī	L(-u)	L(u+Ī)	L(-u+Ī)
12.41	12.153	- 0.257	12.652	0.242
12.42	12.164	- 0.256	12.662	0.242
12.43	12.174	- 0.256	12.671	0.241
12.44	12.185	- 0.2549	12.681	o.241
12.45	12.196	- 0.254	12.690	0.240
12.46	12.206	- 0.254	12.700	0.240
12.47	12.217	- 0.253	12.709	0.239
12.48	12.227	- 0.253	12.719	0.239
12.49	12.238	- 0.252	12.728	0.238
12.50	12.249	- 0.251	12.738	0.238
12.51	12.259	- 0.251	12.747	0.237
12.52	12.270	- 0.250	12.757	0.237
12.53	12.280	- 0.250	12.766	0.236
12.54	12.291	- 0.249	12.775	0.235
12.55	12.302	- 0.248	12.7849	0.2349
12.56	12.312	- 0.248	12.794	0.234
12.57	12.323	- 0.247	12.804	0.234
12.58	12.333	- 0.247	12.813	0.233
12.59	12.344	- 0.246	12.823	0.233
12.60	12.3545	- 0.245	12.832	0.232
12.61	12.365	- 0.2448	12.842	0.232
12.62	12.376	- 0.244	12.851	0.231
12.63	12.386	- 0.244	12.861	0.231
12.64	12.397	- 0.243	12.870	0.230
12.65	12.407	- 0.243	12.880	0.230
12.66	12.418	- 0.242	12.889	0.229
12.67	12.429	- 0.241	12.899	0.229
12.68	12.439	- 0.241	12.908	0.228
12.69	12.450	- 0.240	12.918	0.228
12.70	12.460	- 0.240	12.927	0.227
12.71	12.471	- 0.239	12.937	0.227
12.72	12.481	- 0.239	12.946	0.226
12.73	12.492	- 0.238	12.956	0.226
12.74	12.503	- 0.237	12.965	0.225
12.75	12.513	- 0.237	12.9746	0.2246
12.76	12.524	- 0.236	12.984	0.224
12.77	12.534	- 0.236	12.994	0.224
12.78	12.5447	- 0.235	13.003	0.223
12.79	12.555	- 0.2346	13.013	0.223
12.80	12.566	- 0.234	13.022	0.222

Table 10. Tables of L(u) (continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
12.81	12.576	- 0.234	13.032	0.222
12.82	12.587	- 0.233	13.041	0.221
12.83	12.598	- 0.232	13.051	0.221
12.84	12.608	- 0.232	13.060	0.220
12.85	12.619	- 0.231	13.070	0.220
12.86	12.629	- 0.231	13.079	0.219
12.87	12.640	- 0.230	13.089	0.219
12.88	12.650	- 0.230	13.098	0.218
12.89	12.661	- 0.229	13.108	0.218
12.90	12.671	- 0.229	13.117	0.217
12.91	12.682	- 0.228	13.127	0.217
12.92	12.692	- 0.228	13.136	0.216
12.93	12.703	- 0.227	13.146	0.216
12.94	12.714	- 0.226	13.155	0.215
12.95	12.724	- 0.226	13.1647	0.2147
12.96	12.7345	- 0.225	13.174	0.214
12.97	12.745	- 0.2248	13.184	0.214
12.98	12.756	- 0.224	13.193	0.213
12.99	12.766	- 0.224	13.203	0.213
13.00	12.777	- 0.223	13.212	0.212
13.01	12.787	- 0.223	13.222	0.212
13.02	12.798	- 0.222	13.231	0.211
13.03	12.808	- 0.222	13.241	0.211
13.04	12.819	- 0.221	13.250	0.210
13.05	12.829	- 0.221	13.260	0.210
13.06	12.840	- 0.220	13.270	0.210
13.07	12.850	- 0.220	13.279	0.209
13.08	12.861	- 0.219	13.289	0.209
13.09	12.871	- 0.219	13.298	0.208
13.10	12.882	- 0.218	13.308	0.208
13.11	12.892	- 0.218	13.317	0.207
13.12	12.903	- 0.217	13.327	0.207
13.13	12.913	- 0.217	13.336	0.206
13.14	12.924	- 0.216	13.346	0.206
13.15	12.934	- 0.216	13.355	0.205
13.16	12.9449	- 0.215	13.3648	0.2048
13.17	12.955	- 0.2145	13.374	0.204
13.18	12.966	- 0.214	13.384	0.204
13.19	12.976	- 0.214	13.394	0.204
13.20	12.987	- 0.213	13.403	0.203

Table 10. Tables of L(u) (continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
13.21	12.997	- 0.213	13.413	0.203
13.22	13.008	- 0.212	13.422	0.202
13.23	13.018	- 0.212	13.432	0.202
13.24	13.029	- 0.211	13.441	0.201
13.25	13.039	- 0.211	13.451	0.201
13.26	13.050	- 0.210	13.460	0.200
13.27	13.060	- 0.210	13.470	0.200
13.28	13.071	- 0.209	13.479	0.199
13.29	13.081	- 0.209	13.489	0.199
13.30	13.092	- 0.208	13.499	0.199
13.31	13.102	- 0.208	13.508	0.198
13.32	13.113	- 0.207	13.518	0.198
13.33	13.123	- 0.207	13.527	0.197
13.34	13.134	- 0.206	13.537	0.197
13.35	13.144	- 0.206	13.546	0.196
13.36	13.1548	- 0.205	13.556	0.196
13.37	13.165	- 0.2046	13.565	0.195
13.38	13.176	- 0.204	13.5749	0.1949
13.39	13.186	- 0.204	13.5845	0.1945
13.40	13.197	- 0.203	13.594	0.194
13.41	13.207	- 0.203	13.604	0.194
13.42	13.218	- 0.202	13.613	0.193
13.43	13.228	- 0.202	13.623	0.193
13.44	13.239	- 0.201	13.632	0.192
13.45	13.249	- 0.201	13.642	0.192
13.46	13.260	- 0.200	13.652	0.192
13.47	13.270	- 0.200	13.661	0.191
13.48	13.281	- 0.199	13.671	0.191
13.49	13.291	- 0.199	13.680	0.190
13.50	13.302	- 0.198	13.690	0.190
13.51	13.312	- 0.198	13.699	0.189
13.52	13.322	- 0.198	13.709	0.189
13.53	13.333	- 0.197	13.719	0.189
13.54	13.343	- 0.197	13.728	0.188
13.55	13.354	- 0.196	13.738	0.188
13.56	13.364	- 0.196	13.747	0.187
13.57	13.3747	- 0.195	13.757	0.187
13.58	13.385	- 0.1947	13.766	0.186
13.59	13.396	- 0.194	13.776	0.186
13.60	13.406	- 0.194	13.786	0.186

Table 10. Tables of L(u) (continued).

u	L(u) + ½	L(-u)	L(u+½)	L(-u+½)
13.61	13.417	- 0.193	13.795	0.185
13.62	13.427	- 0.193	13.8047	0.1847
13.63	13.438	- 0.192	13.814	0.184
13.64	13.448	- 0.192	13.824	0.184
13.65	13.458	- 0.192	13.833	0.183
13.66	13.469	- 0.191	13.843	0.183
13.67	13.479	- 0.191	13.853	0.183
13.68	13.490	- 0.190	13.862	0.182
13.69	13.500	- 0.190	13.872	0.182
13.70	13.511	- 0.189	13.881	0.181
13.71	13.521	- 0.189	13.891	0.181
13.72	13.532	- 0.188	13.901	0.181
13.73	13.542	- 0.188	13.910	0.180
13.74	13.552	- 0.188	13.920	0.180
13.75	13.563	- 0.187	13.929	0.179
13.76	13.573	- 0.187	13.939	0.179
13.77	13.584	- 0.186	13.949	0.179
13.78	13.594	- 0.186	13.958	0.178
13.79	13.6046	- 0.185	13.968	0.178
13.80	13.615	- 0.1849	13.977	0.177
13.81	13.626	- 0.184	13.987	0.177
13.82	13.636	- 0.184	13.997	0.177
13.83	13.646	- 0.184	14.006	0.176
13.84	13.657	- 0.183	14.016	0.176
13.85	13.667	- 0.183	14.025	0.175
13.86	13.678	- 0.182	14.0349	0.1749
13.87	13.688	- 0.182	14.0445	0.1745
13.88	13.699	- 0.181	14.054	0.174
13.89	13.709	- 0.181	14.064	0.174
13.90	13.719	- 0.181	14.073	0.173
13.91	13.730	- 0.180	14.083	0.173
13.92	13.740	- 0.180	14.093	0.173
13.93	13.751	- 0.179	14.102	0.172
13.94	13.761	- 0.179	14.112	0.172
13.95	13.771	- 0.179	14.121	0.171
13.96	13.782	- 0.178	14.131	0.171
13.97	13.792	- 0.178	14.141	0.171
13.98	13.803	- 0.177	14.150	0.170
13.99	13.813	- 0.177	14.160	0.170
14.00	13.824	- 0.176	14.170	0.170

Table 10. Tables of L(u)(continued).

u	L(u) + ½	L(-u)	L(u+½)	L(-u+½)
14.01	13.834	- 0.176	14.179	0.169
14.02	13.844	- 0.176	14.189	0.169
14.03	13.8548	- 0.175	14.198	0.168
14.04	13.865	- 0.1747	14.208	0.168
14.05	13.876	- 0.174	14.218	0.168
14.06	13.886	- 0.174	14.227	0.167
14.07	13.896	- 0.174	14.237	0.167
14.08	13.907	- 0.173	14.247	0.167
14.09	13.917	- 0.173	14.256	0.166
14.10	13.928	- 0.172	14.266	0.166
14.11	13.938	- 0.172	14.275	0.165
14.12	13.948	- 0.172	14.285	0.165
14.13	13.959	- 0.171	14.2946	0.1646
14.14	13.969	- 0.171	14.304	0.164
14.15	13.980	- 0.170	14.314	0.164
14.16	13.990	- 0.170	14.324	0.164
14.17	14.000	- 0.170	14.333	0.163
14.18	14.011	- 0.169	14.343	0.163
14.19	14.021	- 0.169	14.352	0.162
14.20	14.032	- 0.168	14.362	0.162
14.21	14.042	- 0.168	14.372	0.162
14.22	14.052	- 0.168	14.381	0.161
14.23	14.063	- 0.167	14.391	0.161
14.24	14.073	- 0.167	14.401	0.161
14.25	14.084	- 0.166	14.410	0.160
14.26	14.094	- 0.166	14.420	0.160
14.27	14.104	- 0.166	14.430	0.160
14.28	14.1147	- 0.165	14.439	0.159
14.29	14.125	- 0.1648	14.449	0.159
14.30	14.136	- 0.164	14.458	0.158
14.31	14.146	- 0.164	14.468	0.158
14.32	14.156	- 0.164	14.478	0.158
14.33	14.167	- 0.163	14.487	0.157
14.34	14.177	- 0.163	14.497	0.157
14.35	14.187	- 0.163	14.507	0.157
14.36	14.198	- 0.162	14.516	0.156
14.37	14.208	- 0.162	14.526	0.156
14.38	14.219	- 0.161	14.536	0.156
14.39	14.229	- 0.161	14.545	0.155
14.40	14.239	- 0.161	14.5548	0.1548

Table 10. Tables of L(u)(continued).

u	L(u) + Ɨ	L(-u)	L(u+Ɨ)	L(-u+Ɨ)
14.41	14.250	- 0.160	14.5645	0.1545
14.42	14.260	- 0.160	14.574	0.154
14.43	14.271	- 0.159	14.584	0.154
14.44	14.281	- 0.159	14.593	0.153
14.45	14.291	- 0.159	14.603	0.153
14.46	14.302	- 0.158	14.613	0.153
14.47	14.312	- 0.158	14.622	0.152
14.48	14.322	- 0.158	14.632	0.152
14.49	14.333	- 0.157	14.642	0.152
14.50	14.343	- 0.157	14.651	0.151
14.51	14.353	- 0.157	14.661	0.151
14.52	14.364	- 0.156	14.671	0.151
14.53	14.374	- 0.156	14.680	0.150
14.54	14.3845	- 0.155	14.690	0.150
14.55	14.3949	- 0.155	14.700	0.150
14.56	14.405	- 0.1547	14.709	0.149
14.57	14.416	- 0.154	14.719	0.149
14.58	14.426	- 0.154	14.729	0.149
14.59	14.436	- 0.154	14.738	0.148
14.60	14.447	- 0.153	14.748	0.148
14.61	14.457	- 0.153	14.758	0.148
14.62	14.467	- 0.153	14.767	0.147
14.63	14.478	- 0.152	14.777	0.147
14.64	14.488	- 0.152	14.787	0.147
14.65	14.499	- 0.151	14.796	0.146
14.66	14.509	- 0.151	14.806	0.146
14.67	14.519	- 0.151	14.816	0.146
14.68	14.530	- 0.150	14.825	0.145
14.69	14.540	- 0.150	14.835	0.145
14.70	14.550	- 0.150	14.8447	0.1447
14.71	14.561	- 0.149	14.854	0.144
14.72	14.571	- 0.149	14.864	0.144
14.73	14.581	- 0.149	14.874	0.144
14.74	14.592	- 0.148	14.883	0.143
14.75	14.602	- 0.148	14.893	0.143
14.76	14.612	- 0.148	14.903	0.143
14.77	14.623	- 0.147	14.912	0.142
14.78	14.633	- 0.147	14.922	0.142
14.79	14.643	- 0.147	14.932	0.142
14.80	14.654	- 0.146	14.941	0.141

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
14.81	14.664	- 0.146	14.951	0.141
14.82	14.674	- 0.146	14.961	0.141
14.83	14.6847	- 0.145	14.971	0.141
14.84	14.695	- 0.1448	14.980	0.140
14.85	14.705	- 0.1445	14.990	0.140
14.86	14.716	- 0.144	15.000	0.140
14.87	14.726	- 0.144	15.009	0.139
14.88	14.736	- 0.144	15.019	0.139
14.89	14.747	- 0.143	15.029	0.139
14.90	14.757	- 0.143	15.038	0.138
14.91	14.767	- 0.143	15.048	0.138
14.92	14.778	- 0.142	15.058	0.138
14.93	14.788	- 0.142	15.067	0.137
14.94	14.798	- 0.142	15.077	0.137
14.95	14.809	- 0.141	15.087	o.137
14.96	14.819	- 0.141	15.096	0.136
14.97	14.829	- 0.141	15.106	0.136
14.98	14.840	- 0.140	15.116	0.136
14.99	14.850	- 0.140	15.126	0.136
15.00	14.860	- 0.140	15.135	0.135
15.01	14.871	- 0.139	15.1449	0.1349
15.02	14.881	- 0.139	15.1545	0.1345
15.03	14.891	- 0.139	15.164	0.134
15.04	14.902	- 0.138	15.174	0.134
15.05	14.912	- 0.138	15.184	0.134
15.06	14.922	- 0.138	15.193	0.133
15.07	14.933	- 0.137	15.203	0.133
15.08	14.943	- 0.137	15.213	0.133
15.09	14.953	- 0.137	15.222	0.132
15.10	14.964	- 0.136	15.232	0.132
15.11	14.974	- 0.136	15.242	0.132
15.12	14.984	- 0.136	15.252	0.132
15.13	14.9946	- 0.135	15.261	0.131
15.14	15.0049	- 0.135	15.271	0.131
15.15	15.015	- 0.1347	15.281	0.131
15.16	15.026	- 0.134	15.290	0.130
15.17	15.036	- 0.134	15.300	0.130
15.18	15.046	- 0.134	15.310	0.130
15.19	15.057	- 0.133	15.320	0.130
15.20	15.067	- 0.133	15.329	0.129

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
15.21	15.077	- 0.133	15.339	0.129
15.22	15.087	- 0.133	15.349	0.129
15.23	15.098	- 0.132	15.358	0.128
15.24	15.108	- 0.132	15.368	0.128
15.25	15.118	- 0.132	15.378	0.128
15.26	15.129	- 0.131	15.387	0.127
15.27	15.139	- 0.131	15.397	0.127
15.28	15.149	- 0.131	15.407	0.127
15.29	15.160	- 0.130	15.417	0.127
15.30	15.170	- 0.130	15.426	0.126
15.31	15.180	- 0.130	15.436	0.126
15.32	15.191	- 0.129	15.446	0.126
15.33	15.201	- 0.129	15.455	0.125
15.34	15.211	- 0.129	15.465	0.125
15.35	15.221	- 0.129	15.4748	0.1248
15.36	15.232	- 0.128	15.4846	0.1246
15.37	15.242	- 0.128	15.494	0.124
15.38	15.252	- 0.128	15.504	0.124
15.39	15.263	- 0.127	15.514	0.124
15.40	15.273	- 0.127	15.523	0.123
15.41	15.283	- 0.127	15.533	0.123
15.42	15.293	- 0.127	15.543	0.123
15.43	15.304	- 0.126	15.553	0.123
15.44	15.314	- 0.126	15.562	0.122
15.45	15.324	- 0.126	15.572	0.122
15.46	15.3346	- 0.125	15.582	0.122
15.47	15.3449	- 0.125	15.592	0.122
15.48	15.355	- 0.1247	15.601	0.121
15.49	15.366	- 0.124	15.611	0.121
15.50	15.376	- 0.124	15.621	0.121
15.51	15.386	- 0.124	15.630	0.120
15.52	15.396	- 0.124	15.640	0.120
15.53	15.407	- 0.123	15.650	0.120
15.54	15.417	- 0.123	15.660	0.120
15.55	15.427	- 0.123	15.669	0.119
15.56	15.438	- 0.122	15.679	0.119
15.57	15.448	- 0.122	15.689	0.119
15.58	15.458	- 0.122	15.699	0.119
15.59	15.468	- 0.122	15.708	0.118
15.60	15.479	- 0.121	15.718	0.118

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
15.61	15.489	- 0.121	15.728	0.118
15.62	15.499	- 0.121	15.737	0.117
15.63	15.510	- 0.120	15.747	0.117
15.64	15.520	- 0.120	15.757	0.117
15.65	15.530	- 0.120	15.767	0.117
15.66	15.540	- 0.120	15.776	0.116
15.67	15.551	- 0.119	15.786	0.116
15.68	15.561	- 0.119	15.796	0.116
15.69	15.571	- 0.119	15.806	0.116
15.70	15.582	- 0.118	15.815	0.115
15.71	15.592	- 0.118	15.825	0.115
15.72	15.602	- 0.118	15.8348	0.1148
15.73	15.612	- 0.118	15.8445	0.1145
15.74	15.623	- 0.117	15.854	0.114
15.75	15.633	- 0.117	15.864	0.114
15.76	15.643	- 0.117	15.874	0.114
15.77	15.653	- 0.117	15.884	0.114
15.78	15.664	- 0.116	15.893	0.113
15.79	15.674	- 0.116	15.903	0.113
15.80	15.684	- 0.116	15.913	0.113
15.81	15.6945	- 0.115	15.922	0.112
15.82	15.7047	- 0.115	15.932	0.112
15.83	15.715	- 0.1149	15.942	0.112
15.84	15.725	- 0.1146	15.952	0.112
15.85	15.736	- 0.114	15.961	0.111
15.86	15.746	- 0.114	15.971	0.111
15.87	15.756	- 0.114	15.981	0.111
15.88	15.766	- 0.114	15.991	0.111
15.89	15.777	- 0.113	16.000	0.110
15.90	15.787	- 0.113	16.010	0.110
15.91	15.797	- 0.113	16.020	0.110
15.92	15.807	- 0.113	16.030	0.110
15.93	15.818	- 0.112	16.039	0.109
15.94	15.828	- 0.112	16.049	0.109
15.95	15.838	- 0.112	16.059	0.109
15.96	15.848	- 0.112	16.069	0.109
15.97	15.859	- 0.111	16.078	0.108
15.98	15.869	- 0.111	16.088	0.108
15.99	15.879	- 0.111	16.098	0.108
16.00	15.890	- 0.110	16.108	0.108

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
16.01	15.900	- 0.110	16.117	0.107
16.02	15.910	- 0.110	16.127	0.107
16.03	15.920	- 0.110	16.137	0.107
16.04	15.931	- 0.109	16.147	0.107
16.05	15.941	- 0.109	16.157	0.107
16.06	15.951	- 0.109	16.166	0.106
16.07	15.961	- 0.109	16.176	0.106
16.08	15.972	- 0.108	16.186	0.106
16.09	15.982	- 0.108	16.196	0.106
16.10	15.992	- 0.108	16.205	0.105
16.11	16.002	- 0.108	16.215	0.105
16.12	16.013	- 0.107	16.2248	0.1048
16.13	16.023	- 0.107	16.2346	0.1046
16.14	16.033	- 0.107	16.244	0.104
16.15	16.043	- 0.107	16.254	0.104
16.16	16.054	- 0.106	16.264	0.104
16.17	16.064	- 0.106	16.274	0.104
16.18	16.074	- 0.106	16.283	0.103
16.19	16.084	- 0.106	16.293	0.103
16.20	16.0945	- 0.105	16.303	0.103
16.21	16.1047	- 0.105	16.313	0.103
16.22	16.115	- 0.1049	16.322	0.102
16.23	16.125	- 0.1047	16.332	-.102
16.24	16.136	- 0.104	16.342	0.102
16.25	16.146	- 0.104	16.352	0.102
16.26	16.156	- 0.104	16.362	0.102
16.27	16.166	- 0.104	16.371	0.101
16.28	16.176	- 0.104	16.381	0.101
16.29	16.187	- 0.103	16.391	0.101
16.30	16.197	- 0.103	16.401	0.101
16.31	16.207	- 0.103	16.410	0.100
16.32	16.217	- 0.103	16.420	0.100
16.33	16.228	- 0.102	16.430	0.100
16.34	16.238	- 0.102	16.440	0.100
16.35	16.248	- 0.102	16.449	0.099
16.36	16.258	- 0.102	16.459	0.099
16.37	16.269	- 0.101	16.469	0.099
16.38	16.279	- 0.101	16.479	0.099
16.39	16.289	- 0.101	16.489	0.099
16.40	16.299	- 0.101	16.498	0.098

Table 10. Tables of L(u) (continued).

u	L(u) + Ɨ	L(-u)	L(u+Ɨ)	L(-u+Ɨ)
16.41	16.310	- 0.100	16.508	0.098
16.42	16.320	- 0.100	16.518	0.098
16.43	16.330	- 0.100	16.528	0.098
16.44	16.340	- 0.100	16.537	0.097
16.45	16.351	- 0.099	16.547	0.097
16.46	16.361	- 0.099	16.557	0.097
16.47	16.371	- 0.099	16.567	0.097
16.48	16.381	- 0.099	16.577	0.097
16.49	16.391	- 0.099	16.586	0.096
16.50	16.402	- 0.098	16.596	0.096
16.51	16.412	- 0.098	16.606	0.096
16.52	16.422	- 0.098	16.616	0.096
16.53	16.432	- 0.098	16.625	0.095
16.54	16.443	- 0.097	16.635	0.095
16.55	16.453	- 0.097	16.645	0.095
16.56	16.463	- 0.097	16.6548	0.0948
16.57	16.473	- 0.097	16.6646	0.0946
16.58	16.483	- 0.097	16.674	0.094
16.59	16.494	- 0.096	16.684	0.094
16.60	16.504	- 0.096	16.694	0.094
16.61	16.514	- 0.096	16.704	0.094
16.62	16.524	- 0.096	16.714	0.094
16.63	16.5346	- 0.095	16.723	0.093
16.64	16.5448	- 0.095	16.733	0.093
16.65	16.555	- 0.0949	16.743	0.093
16.66	16.565	- 0.0947	16.753	0.093
16.67	16.575	- 0.0945	16.763	0.093
16.68	16.586	- 0.094	16.772	0.092
16.69	16.596	- 0.094	16.782	0.092
16.70	16.606	- 0.094	16.792	0.092
16.71	16.616	- 0.094	16.802	0.092
16.72	16.627	- 0.093	16.811	0.091
16.73	16.637	- 0.093	16.821	0.091
16.74	16.647	- 0.093	16.831	0.091
16.75	16.657	- 0.093	16.841	0.091
16.76	16.667	- 0.093	16.851	0.091
16.77	16.678	- 0.092	16.860	0.090
16.78	16.688	- 0.092	16.870	0.090
16.79	16.698	- 0.092	16.880	0.090
16.80	16.708	- 0.092	16.890	0.090

Table 10. Tables of L(u) (continued).

u	L(u) + Ξ	L(-u)	L(u+Ξ)	L(-u+Ξ)
16.81	16.719	- 0.091	16.900	0.090
16.82	16.729	- 0.091	16.909	0.089
16.83	16.739	- 0.091	16.919	0.089
16.84	16.749	- 0.091	16.929	0.089
16.85	16.759	- 0.091	16.939	0.089
16.86	16.770	- 0.090	16.949	0.089
16.87	16.780	- 0.090	16.958	0.088
16.88	16.790	- 0.090	16.968	0.088
16.89	16.800	- 0.090	16.978	0.088
16.90	16.810	- 0.090	16.988	0.088
16.91	16.821	- 0.089	16.998	0.088
16.92	16.831	- 0.089	17.007	0.087
16.93	16.841	- 0.089	17.017	0.087
16.94	16.851	- 0.089	17.027	0.087
16.95	16.861	- 0.089	17.037	0.087
16.96	16.872	- 0.088	17.047	0.087
16.97	16.882	- 0.088	17.056	0.086
16.98	16.892	- 0.088	17.066	0.086
16.99	16.902	- 0.088	17.076	0.086
17.00	16.912	- 0.088	17.086	0.086
17.01	16.923	- 0.087	17.096	0.086
17.02	16.933	- 0.087	17.105	0.085
17.03	16.943	- 0.087	17.115	0.085
17.04	16.953	- 0.087	17.125	0.085
17.05	16.963	- 0.087	17.1348	0.0848
17.06	16.974	- 0.086	17.1446	0.0846
17.07	16.984	- 0.086	17.154	0.084
17.08	16.994	- 0.086	17.164	0.084
17.09	17.004	- 0.086	17.174	0.084
17.10	17.014	- 0.086	17.184	0.084
17.11	17.0246	- 0.085	17.194	0.084
17.12	17.0348	- 0.085	17.203	0.083
17.13	17.045	- 0.0849	17.213	0.083
17.14	17.055	- 0.0847	17.223	0.083
17.15	17.065	- 0.0845	17.233	0.083
17.16	17.076	- 0.084	17.243	0.083
17.17	17.086	- 0.084	17.253	0.083
17.18	17.096	- 0.084	17.262	0.082
17.19	17.106	- 0.084	17.272	0.082
17.20	17.116	- 0.084	17.282	0.082

Table 10. Tables of L(u)(continued).

u	L(u) + ½	L(-u)	L(u+½)	L(-u+½)
17.21	17.127	- 0.083	17.292	0.082
17.22	17.137	- 0.083	17.302	0.082
17.23	17.147	- 0.083	17.311	0.081
17.24	17.157	- 0.083	17.321	0.081
17.25	17.167	- 0.083	17.331	0.081
17.26	17.178	- 0.082	17.341	0.081
17.27	17.188	- 0.082	17.351	0.081
17.28	17.198	- 0.082	17.360	0.080
17.29	17.208	- 0.082	17.370	0.080
17.30	17.218	- 0.082	17.380	0.080
17.31	17.229	- 0.081	17.390	0.080
17.32	17.239	- 0.081	17.400	0.080
17.33	17.249	- 0.081	17.410	0.080
17.34	17.259	- 0.081	17.419	0.079
17.35	17.269	- 0.081	17.429	0.079
17.36	17.279	- 0.081	17.439	0.079
17.37	17.290	- 0.080	17.449	0.079
17.38	17.300	- 0.080	17.459	0.079
17.39	17.310	- 0.080	17.468	0.078
17.40	17.320	- 0.080	17.478	0.078
17.41	17.330	- 0.080	17.488	0.078
17.42	17.341	- 0.079	17.498	0.078
17.43	17.351	- 0.079	17.508	0.078
17.44	17.361	- 0.079	17.518	0.078
17.45	17.371	- 0.079	17.527	0.077
17.46	17.381	- 0.079	17.537	0.077
17.47	17.392	- 0.078	17.547	0.077
17.48	17.402	- 0.078	17.557	0.077
17.49	17.412	- 0.078	17.567	0.077
17.50	17.422	- 0.078	17.577	0.077
17.51	17.432	- 0.078	17.586	0.076
17.52	17.442	- 0.078	17.596	0.076
17.53	17.453	- 0.077	17.606	0.076
17.54	17.463	- 0.077	17.616	0.076
17.55	17.473	- 0.077	17.626	0.076
17.56	17.483	- 0.077	17.636	0.076
17.57	17.493	- 0.077	17.645	0.075
17.58	17.504	- 0.076	17.655	0.075
17.59	17.514	- 0.076	17.6649	0.0749
17.60	17.524	- 0.076	17.6748	0.0748

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
17.61	17.534	- 0.076	17.6846	0.0746
17.62	17.544	- 0.076	17.694	0.074
17.63	17.554	- 0.076	17.704	0.074
17.64	17.5645	- 0.075	17.714	0.074
17.65	17.5747	- 0.075	17.724	0.074
17.66	17.5849	- 0.075	17.734	0.074
17.67	17.595	- 0.0749	17.744	0.074
17.68	17.605	- 0.0747	17.753	0.073
17.69	17.615	- 0.0745	17.763	0.073
17.70	17.626	- 0.074	17.773	0.073
17.71	17.636	- 0.074	17.783	0.073
17.72	17.646	- 0.074	17.793	0.073
17.73	17.656	- 0.074	17.803	0.073
17.74	17.666	- 0.074	17.812	0.072
17.75	17.676	- 0.074	17.822	0.072
17.76	17.687	- 0.073	17.832	0.072
17.77	17.697	- 0.073	17.842	0.072
17.78	17.707	- 0.073	17.852	0.072
17.79	17.717	- 0.073	17.862	0.072
17.80	17.727	- 0.073	17.871	0.071
17.81	17.737	- 0.073	17.881	0.071
17.82	17.748	- 0.072	17.891	0.071
17.83	17.758	- 0.072	17.901	0.071
17.84	17.768	- 0.072	17.911	0.071
17.85	17.778	- 0.072	17.921	0.071
17.86	17.788	- 0.072	17.931	0.071
17.87	17.798	- 0.072	17.940	0.070
17.88	17.809	- 0.071	17.950	0.070
17.89	17.819	- 0.071	17.960	0.070
17.90	17.829	- 0.071	17.970	0.070
17.91	17.839	- 0.071	17.980	0.070
17.92	17.849	- 0.071	17.990	0.070
17.93	17.859	- 0.071	17.999	0.069
17.94	17.870	- 0.070	18.009	0.069
17.95	17.880	- 0.070	18.019	0.069
17.96	17.890	- 0.070	18.029	0.069
17.97	17.900	- 0.070	18.039	0.069
17.98	17.910	- 0.070	18.049	0.069
17.99	17.920	- 0.070	18.058	0.068
18.00	17.931	- 0.069	18.068	0.068

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
18.01	17.941	- 0.069	18.078	0.068
18.02	17.951	- 0.069	18.088	0.068
18.03	17.961	- 0.069	18.098	0.068
18.04	17.971	- 0.069	18.108	0.068
18.05	17.981	- 0.069	18.118	0.068
18.06	17.992	- 0.068	18.127	0.067
18.07	18.002	- 0.068	18.137	0.067
18.08	18.012	- 0.068	18.147	0.067
18.09	18.022	- 0.068	18.157	0.067
18.10	18.032	- 0.068	18.167	0.067
18.11	18.042	- 0.068	18.177	0.067
18.12	18.053	- 0.067	18.186	0.066
18.13	18.063	- 0.067	18.196	0.066
18.14	18.073	- 0.067	18.206	0.066
18.15	18.083	- 0.067	18.216	0.066
18.16	18.093	- 0.067	18.226	0.066
18.17	18.103	- 0.067	18.236	0.066
18.18	18.113	- 0.067	18.246	0.066
18.19	18.124	- 0.066	18.255	0.065
18.20	18.134	- 0.066	18.265	0.065
18.21	18.144	- 0.066	18.275	0.065
18.22	18.154	- 0.066	18.2849	0.0649
18.23	18.164	- 0.066	18.2947	0.0647
18.24	18.174	- 0.066	18.3046	0.0646
18.25	18.1845	- 0.065	18.314	0.064
18.26	18.1946	- 0.065	18.324	0.064
18.27	18.2048	- 0.065	18.334	0.064
18.28	18.2149	- 0.065	18.344	0.064
18.29	18.225	- 0.0648	18.354	0.064
18.30	18.235	- 0.0647	18.364	0.064
18.31	18.245	- 0.0645	18.374	0.074
18.32	18.256	- 0.064	18.383	0.063
18.33	18.266	- 0.064	18.393	0.063
18.34	18.276	- 0.064	18.403	0.063
18.35	18.286	- 0.064	18.413	0.063
18.36	18.296	- 0.064	18.423	0.063
18.37	18.306	- 0.064	18.433	0.063
18.38	18.316	- 0.064	18.443	0.063
18.39	18.327	- 0.063	18.452	0.062
18.40	18.337	- 0.063	18.462	0.062

Table 10. Tables of L(u) (continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
18.41	18.347	- 0.063	18.472	0.062
18.42	18.357	- 0.063	18.482	0.062
18.43	18.367	- 0.063	18.492	0.062
18.44	18.377	- 0.063	18.502	0.062
18.45	18.387	- 0.063	18.512	0.062
18.46	18.398	- 0.062	18.521	0.061
18.47	18.408	- 0.062	18.531	0.061
18.48	18.418	- 0.062	18.541	0.061
18.49	18.428	- 0.062	18.551	0.061
18.50	18.438	- 0.062	18.561	0.061
18.51	18.448	- 0.062	18.571	0.061
18.52	18.459	- 0.061	18.581	0.061
18.53	18.469	- 0.061	18.590	0.060
18.54	18.479	- 0.061	18.600	0.060
18.55	18.489	- 0.061	18.610	0.060
18.56	18.499	- 0.061	18.620	0.060
18.57	18.509	- 0.061	18.630	0.060
18.58	18.519	- 0.061	18.640	0.060
18.59	18.529	- 0.061	18.650	0.060
18.60	18.540	- 0.060	18.660	0.060
18.61	18.550	- 0.060	18.669	0.059
18.62	18.560	- 0.060	18.679	0.059
18.63	18.570	- 0.060	18.689	0.059
18.64	18.580	- 0.060	18.699	0.059
18.65	18.590	- 0.060	18.709	0.059
18.66	18.600	- 0.060	18.719	0.059
18.67	18.611	- 0.059	18.729	0.059
18.68	18.621	- 0.059	18.738	0.058
18.69	18.631	- 0.059	18.748	0.058
18.70	18.641	- 0.059	18.758	0.058
18.71	18.651	- 0.059	18.768	0.058
18.72	18.661	- 0.059	18.778	0.058
18.73	18.671	- 0.059	18.788	0.058
18.74	18.682	- 0.058	18.798	0.058
18.75	18.692	- 0.058	18.808	0.058
18.76	18.702	- 0.058	18.817	0.057
18.77	18.712	- 0.058	18.827	0.057
18.78	18.722	- 0.058	18.837	0.057
18.79	18.732	- 0.058	18.847	0.057
18.80	18.742	- 0.058	18.857	0.057

Table 10. Tables of L(u)(continued).

u	L(u) + Ī	L(-u)	L(u+Ī)	L(-u+Ī)
18.81	18.753	- 0.057	18.867	0.057
18.82	18.763	- 0.057	18.877	0.057
18.83	18.773	- 0.057	18.886	0.056
18.84	18.783	- 0.057	18.896	0.056
18.85	18.793	- 0.057	18.906	0.056
18.86	18.803	- 0.057	18.916	0.056
18.87	18.813	- 0.057	18.926	0.056
18.88	18.823	- 0.057	18.936	0.056
18.89	18.834	- 0.056	18.946	0.056
18.90	18.844	- 0.056	18.956	0.056
18.91	18.854	- 0.056	18.965	0.055
18.92	18.864	- 0.056	18.975	0.055
18.93	18.874	- 0.056	18.985	0.055
18.94	18.884	- 0.056	18.995	0.055
18.95	18.894	- 0.056	19.0049	0.0549
18.96	18.904	- 0.056	19.0148	0.0548
18.97	18.9145	- 0.055	19.0247	0.0547
18.98	18.9247	- 0.055	19.0345	0.0545
18.99	18.9348	- 0.055	19.044	0.054
19.00	18.9449	- 0.055	19.054	0.054
19.01	18.955	- 0.0548	19.064	0.054
19.02	18.965	- 0.0547	19.074	0.054
19.03	18.975	- 0.0546	19.084	0.054
19.04	18.985	- 0.0545	19.094	0.054
19.05	18.996	- 0.054	19.104	0.054
19.06	19.006	- 0.054	19.114	0.054
19.07	19.016	- 0.054	19.123	0.053
19.08	19.026	- 0.054	19.133	0.053
19.09	19.036	- 0.054	19.143	0.053
19.10	19.046	- 0.054	19.153	0.053
19.11	19.056	- 0.054	19.163	0.053
19.12	19.066	- 0.054	19.173	0.053
19.13	19.077	- 0.053	19.183	0.053
19.14	19.087	- 0.053	19.193	0.053
19.15	19.097	- 0.053	19.203	0.053
19.16	19.107	- 0.053	19.212	0.052
19.17	19.117	- 0.053	19.222	0.052
19.18	19.127	- 0.053	19.232	0.052
19.19	19.137	- 0.053	19.242	0.052
19.20	19.147	- 0.053	19.252	0.052

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
19.21	19.158	- 0.052	19.262	0.052
19.22	19.168	- 0.052	19.272	0.052
19.23	19.178	- 0.052	19.282	0.052
19.24	19.188	- 0.052	19.291	0.051
19.25	19.198	- 0.052	19.301	0.051
19.26	19.208	- 0.052	19.311	0.051
19.27	19.218	- 0.052	19.321	0.051
19.28	19.228	- 0.052	19.331	0.051
19.29	19.239	- 0.051	19.341	0.051
19.30	19.249	- 0.051	19.351	0.051
19.31	19.259	- 0.051	19.361	0.051
19.32	19.269	- 0.051	19.370	0.050
19.33	19.279	- 0.051	19.380	0.050
19.34	19.289	- 0.051	19.390	0.050
19.35	19.299	- 0.051	19.400	0.050
19.36	19.309	- 0.051	19.410	0.050
19.37	19.319	- 0.051	19.420	0.050
19.38	19.330	- 0.050	19.430	0.050
19.39	19.340	- 0.050	19.440	0.050
19.40	19.350	- 0.050	19.450	0.050
19.41	19.360	- 0.050	19.459	0.049
19.42	19.370	- 0.050	19.469	0.049
19.43	19.380	- 0.050	19.479	0.049
19.44	19.390	- 0.050	19.489	0.049
19.45	19.400	- 0.050	19.499	0.049
19.46	19.411	- 0.049	19.509	0.049
19.47	19.421	- 0.049	19.519	0.049
19.48	19.431	- 0.049	19.529	0.049
19.49	19.441	- 0.049	19.539	0.049
19.50	19.451	- 0.049	19.548	0.048
19.51	19.461	- 0.049	19.558	0.048
19.52	19.471	- 0.049	19.568	0.048
19.53	19.481	- 0.049	19.578	0.048
19.54	19.491	- 0.049	19.588	0.048
19.55	19.502	- 0.048	19.598	0.048
19.56	19.512	- 0.048	19.608	0.048
19.57	19.522	- 0.048	19.618	0.048
19.58	19.532	- 0.048	19.628	0.048
19.59	19.542	- 0.048	19.637	0.047
19.60	19.552	- 0.048	19.647	0.047

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
19.61	19.562	- 0.048	19.657	0.047
19.62	19.572	- 0.048	19.667	0.047
19.63	19.582	- 0.048	19.677	0.047
19.64	19.593	- 0.047	19.687	0.047
19.65	19.603	- 0.047	19.687	0.047
19.66	19.613	- 0.047	19.707	0.047
19.67	19.623	- 0.047	19.717	0.047
19.68	19.633	- 0.047	19.727	0.047
19.69	19.643	- 0.047	19.736	0.046
19.70	19.653	- 0.047	19.746	0.046
19.71	19.663	- 0.047	19.756	0.046
19.72	19.673	- 0.047	19.766	0.046
19.73	19.684	- 0.046	19.776	0.046
19.74	19.694	- 0.046	19.786	0.046
19.75	19.704	- 0.046	19.796	0.046
19.76	19.714	- 0.046	19.806	0.046
19.77	19.724	- 0.046	19.816	0.046
19.78	19.734	- 0.046	19.825	0.045
19.79	19.744	- 0.046	19.835	0.045
19.80	19.754	- 0.046	19.845	0.045
19.81	19.764	- 0.046	19.855	0.045
19.82	19.774	- 0.046	19.865	0.045
19.83	19.7846	- 0.045	19.8749	0.0449
19.84	19.7947	- 0.045	19.8848	0.0448
19.85	19.8048	- 0.045	19.8947	0.0447
19.86	19.8149	- 0.045	19.9046	0.0446
19.87	19.825	- 0.0449	19.9145	0.0445
19.88	19.835	- 0.0448	19.924	0.044
19.89	19.845	- 0.0447	19.934	0.044
19.90	19.855	- 0.0446	19.944	0.044
19.91	19.865	- 0.0445	19.954	0.044
19.92	19.876	- 0.044	19.964	0.044
19.93	19.886	- 0.044	19.974	0.044
19.94	19.896	- 0.044	19.984	0.044
19.95	19.906	- 0.044	19.994	0.044
19.96	19.916	- 0.044	20.004	0.044
19.97	19.926	- 0.044	20.014	0.044
19.98	19.936	- 0.044	20.023	0.043
19.99	19.946	- 0.044	20.033	0.043
20.00	19.956	- 0.044	20.043	0.043

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
20.01	19.966	- 0.044	20.053	0.043
20.02	19.977	- 0.043	20.063	0.043
20.03	19.987	- 0.043	20.073	0.043
20.04	19.997	- 0.043	20.083	0.043
20.05	20.007	- 0.043	20.093	0.043
20.06	20.017	- 0.043	20.103	0.043
20.07	20.027	- 0.043	20.113	0.043
20.08	20.037	- 0.043	20.122	0.042
20.09	20.047	- 0.043	20.132	0.042
20.10	20.057	- 0.043	20.142	0.042
20.11	20.067	- 0.043	20.152	0.042
20.12	20.078	- 0.042	20.162	0.042
20.13	20.088	- 0.042	20.172	0.042
20.14	20.098	- 0.042	20.182	0.042
20.15	20.108	- 0.042	20.192	0.042
20.16	20.118	- 0.042	20.202	0.042
20.17	20.128	- 0.042	20.212	0.042
20.18	20.138	- 0.042	20.221	0.041
20.19	20.148	- 0.042	20.231	0.041
20.20	20.158	- 0.042	20.241	0.041
20.21	20.168	- 0.042	20.251	0.041
20.22	20.179	- 0.041	20.261	0.041
20.23	20.189	- 0.041	20.271	0.041
20.24	20.199	- 0.041	20.281	0.041
20.25	20.209	- 0.041	20.291	0.041
20.26	20.219	- 0.041	20.301	0.041
20.27	20.229	- 0.041	20.311	0.041
20.28	20.239	- 0.041	20.321	0.041
20.29	20.249	- 0.041	20.330	0.040
20.30	20.259	- 0.041	20.340	0.040
20.31	20.269	- 0.041	20.350	0.040
20.32	20.279	- 0.041	20.360	0.040
20.33	20.290	- 0.040	20.370	0.040
20.34	20.300	- 0.040	20.380	0.040
20.35	20.310	- 0.040	20.390	0.040
20.36	20.320	- 0.040	20.400	0.040
20.37	20.320	- 0.040	20.410	0.040
20.38	20.340	- 0.040	20.420	0.040
20.39	20.350	- 0.040	20.430	0.040
20.40	20.360	- 0.040	20.439	0.039

Table 10. Tables of L(u)(continued).

u	L(u) + ½	L(-u)	L(u+½)	L(-u+½)
20.41	20.370	- 0.040	20.449	0.039
20.42	20.380	- 0.040	20.459	0.039
20.43	20.390	- 0.040	20.469	0.039
20.44	20.401	- 0.039	20.479	0.039
20.45	20.411	- 0.039	20.489	0.039
20.46	20.421	- 0.039	20.499	0.039
20.47	20.431	- 0.039	20.509	0.039
20.48	20.441	- 0.039	20.519	0.039
20.49	20.451	- 0.039	20.529	0.039
20.50	20.461	- 0.039	20.539	0.039
20.51	20.471	- 0.039	20.548	0.038
20.52	20.481	- 0.039	20.558	0.038
20.53	20.491	- 0.039	20.568	0.038
20.54	20.501	- 0.039	20.578	0.038
20.55	20.512	- 0.038	20.588	0.038
20.56	20.522	- 0.038	20.598	0.038
20.57	20.532	- 0.038	20.608	0.038
20.58	20.542	- 0.038	20.618	0.038
20.59	20.552	- 0.038	20.628	0.038
20.60	20.562	- 0.038	20.638	0.038
20.61	20.572	- 0.038	20.648	0.038
20.62	20.582	- 0.038	20.657	0.037
20.63	20.592	- 0.038	20.667	0.037
20.64	20.602	- 0.038	20.677	0.037
20.65	20.612	- 0.038	20.687	0.037
20.66	20.623	- 0.037	20.697	0.037
20.67	20.633	- 0.037	20.707	0.037
20.68	20.643	- 0.037	20.717	0.037
20.69	20.653	- 0.037	20.727	0.037
20.70	20.663	- 0.037	20.737	0.037
20.71	20.673	- 0.037	20.747	0.037
20.72	20.683	- 0.037	20.757	0.037
20.73	20.693	- 0.037	20.767	0.037
20.74	20.703	- 0.037	20.776	0.036
20.75	20.713	- 0.037	20.786	0.036
20.76	20.723	- 0.037	20.796	0.036
20.77	20.733	- 0.037	20.806	0.036
20.78	20.744	- 0.036	20.816	0.036
20.79	20.754	- 0.036	20.826	0.036
20.80	20.764	- 0.036	20.836	0.036

Table 10. Tables of L(u)(continued).

u	L(u) + ½	L(-u)	L(u+½)	L(-u+½)
20.81	20.774	- 0.036	20.846	0.036
20.82	20.784	- 0.036	20.856	0.036
20.83	20.794	- 0.036	20.866	0.036
20.84	20.804	- 0.036	20.876	0.036
20.85	20.814	- 0.036	20.886	0.036
20.86	20.824	- 0.036	20.895	0.035
20.87	20.834	- 0.036	20.905	0.035
20.88	20.844	- 0.036	20.915	0.035
20.89	20.854	- 0.036	20.925	0.035
20.90	20.864	- 0.036	20.935	0.035
20.91	20.8746	- 0.035	20.945	0.035
20.92	20.8847	- 0.035	20.9549	0.0349
20.93	20.8948	- 0.035	20.9649	0.0349
20.94	20.9048	- 0.035	20.9748	0.0348
20.95	20.9149	- 0.035	20.9847	0.0347
20.96	20.925	- 0.0349	20.9946	0.0346
20.97	20.935	- 0.0348	21.0045	0.0345
20.98	20.945	- 0.0347	21.0145	0.0345
20.99	20.955	- 0.0347	21.024	0.034
21.00	20.965	- 0.0346	21.034	0.034
21.01	20.975	- 0.0345	21.044	0.034
21.02	20.986	- 0.034	21.054	0.034
21.03	20.996	- 0.034	21.064	0.034
21.04	21.006	- 0.034	21.074	0.034
21.05	21.016	- 0.034	21.084	0.034
21.06	21.026	- 0.034	21.094	0.034
21.07	21.036	- 0.034	21.104	0.034
21.08	21.046	- 0.034	21.114	0.034
21.09	21.056	- 0.034	21.124	0.034
21.10	21.066	- 0.034	21.134	0.034
21.11	21.076	- 0.034	21.144	0 034
21.12	21.086	- 0.034	21.153	0.033
21.13	21.096	- 0.034	21.163	0.033
21.14	21.106	- 0.034	21.173	0.033
21.15	21.117	- 0.033	21.183	0.033
21.16	21.127	- 0.033	21.193	0.033
21.17	21.137	- 0.033	21.203	0.033
21.18	21.147	- 0.033	21.213	0.033
21.19	21.157	- 0.033	21.223	0.033
21.20	21.167	- 0.033	21.233	0.033

Table 10. Tables of L(u)(continued).

u	L(u) + Ξ	L(-u)	L(u+Ξ)	L(-u+Ξ)
21.21	21.177	- 0.033	21.243	0.033
21.22	21.187	- 0.033	21.253	0.033
21.23	21.197	- 0.033	21.263	0.033
21.24	21.207	- 0.033	21.273	0.033
21.25	21.217	- 0.033	21.282	0.032
21.26	21.227	- 0.033	21.292	0.032
21.27	21.237	- 0.033	21.302	0.032
21.28	21.248	- 0.032	21.312	0.032
21.29	21.258	- 0.032	21.322	0.032
21.30	21.268	- 0.032	21.332	0.032
21.31	21.278	- 0.032	21.342	0.032
21.32	21.288	- 0.032	21.352	0.032
21.33	21.298	- 0.032	21.362	0.032
21.34	21.308	- 0.032	21.372	0.032
21.35	21.318	- 0.032	21.382	0.032
21.36	21.328	- 0.032	21.392	0.032
21.37	21.338	- 0.032	21.402	0.032
21.38	21.348	- 0.032	21.411	0.031
21.39	21.358	- 0.032	21.421	0.031
21.40	21.368	- 0.032	21.431	0.031
21.41	21.378	- 0.032	21.441	0.031
21.42	21.389	- 0.031	21.451	0.031
21.43	21.399	- 0.031	21.461	0.031
21.44	21.409	- 0.031	21.471	0.031
21.45	21.419	- 0.031	21.481	0.031
21.46	21.429	- 0.031	21.491	0.031
21.47	21.439	- 0.031	21.501	0.031
21.48	21.449	- 0.031	21.511	0.031
21.49	21.459	- 0.031	21.521	0.031
21.50	21.469	- 0.031	21.531	0.031
21.51	21.479	- 0.031	21.541	0.031
21.52	21.489	- 0.031	21.550	0.030
21.53	21.499	- 0.031	21.560	0.030
21.54	21.509	- 0.031	21.570	0.030
21.55	21.519	- 0.031	21.580	0.030
21.56	21.530	- 0.030	21.590	0.030
21.57	21.540	- 0.030	21.600	0.030
21.58	21.550	- 0.030	21.610	0.030
21.59	21.560	- 0.030	21.620	0.030
21.60	21.570	- 0.030	21.630	0.030

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
21.61	21.580	- 0.030	21.640	0.030
21.62	21.590	- 0.030	21.650	0.030
21.63	21.600	- 0.030	21.670	0.030
21.64	21.610	- 0.030	21.670	0.030
21.65	21.620	- 0.030	21.680	0.030
21.66	21.630	- 0.030	21.690	0.030
21.67	21.640	- 0.030	21.699	0.029
21.68	21.650	- 0.030	21.709	0.029
21.69	21.660	- 0.030	21.719	0.029
21.70	21.671	- 0.029	21.729	0.029
21.71	21.681	- 0.029	21.739	0.029
21.72	21.691	- 0.029	21.749	0.029
21.73	21.701	- 0.029	21.759	0.029
21.74	21.711	- 0.029	21.769	0.029
21.75	21.721	- 0.029	21.779	0.029
21.76	21.731	- 0.029	21.789	0.029
21.77	21.741	- 0.029	21.799	0.029
21.78	21.751	- 0.029	21.809	0.029
21.79	21.761	- 0.029	21.819	0.029
21.80	21.771	- 0.029	21.829	0.029
21.81	21.781	- 0.029	21.839	0.029
21.82	21.791	- 0.029	21.848	0.028
21.83	21.801	- 0.029	21.858	0.028
21.84	21.811	- 0.029	21.868	0.028
21.85	21.822	- 0.028	21.878	0.028
21.86	21.832	- 0.028	21.888	0.028
21.87	21.842	- 0.028	21.898	0.028
21.88	21.852	- 0.028	21.908	0.028
21.89	21.862	- 0.028	21.918	0.028
21.90	21.872	- 0.028	21.928	0.028
21.91	21.882	- 0.028	21.938	0.028
21.92	21.892	- 0.028	21.948	0.028
21.93	21.902	- 0.028	21.958	0.028
21.94	21.912	- 0.028	21.968	0.028
21.95	21.922	- 0.028	21.978	0.028
21.96	21.932	- 0.028	21.988	0.028
21.97	21.942	- 0.028	21.998	0.028
21.98	21.952	- 0.028	22.007	0.027
21.99	21.962	- 0.028	22.017	0.027
22.00	21.973	- 0.027	22.027	0.027

Table 10. Tables of L(u)(continued).

u	L(u) + Ɨ	L(-u)	L(u+Ɨ)	L(-u+Ɨ)
22.01	21.973	- 0.027	22.037	0.027
22.02	21.993	- 0.027	22.047	0.027
22.03	22.003	- 0.027	22.057	0.027
22.04	22.013	- 0.027	22.067	0.027
22.05	22.023	- 0.027	22.077	0.027
22.06	22.033	- 0.027	22.087	0.027
22.07	22.043	- 0.027	22.097	0.027
22.08	22.053	- 0.027	22.107	0.027
22.09	22.063	- 0.027	22.117	0.027
22.10	22.073	- 0.027	22.127	0.027
22.11	22.083	- 0.027	22.137	0.027
22.12	22.093	- 0.027	22.147	0.027
22.13	22.103	- 0.027	22.157	0.027
22.14	22.113	- 0.027	22.166	0.026
22.15	22.123	- 0.027	22.176	0.026
22.16	22.134	- 0.026	22.186	0.026
22.17	22.144	- 0.026	22.196	0.026
22.18	22.154	- 0.026	22.206	0.026
22.19	22.164	- 0.026	22.216	0.026
22.20	22.174	- 0.026	22.226	0.026
22.21	22.184	- 0.026	22.236	0.026
22.22	22.194	- 0.026	22.246	0.026
22.23	22.204	- 0.026	22.256	0.026
22.24	22.214	- 0.026	22.266	0.026
22.25	22.224	- 0.026	22.276	0.026
22.26	22.234	- 0.026	22.286	0.026
22.27	22.244	- 0.026	22.296	0.026
22.28	22.254	- 0.026	22.306	0.026
22.29	22.264	- 0.026	22.316	0.026
22.30	22.274	- 0.026	22.325	0.025
22.31	22.284	- 0.026	22.335	0.025
22.32	22.294	- 0.026	22.345	0.025
22.33	22.3045	- 0.025	22.355	0.025
22.34	22.3145	- 0.025	22.365	0.025
22.35	22.3246	- 0.025	22.375	0.025
22.36	22.3347	- 0.025	22.385	0.025
22.37	22.3447	- 0.025	22.395	0.025
22.38	22.3548	- 0.025	22.405	0.025
22.39	22.3648	- 0.025	22.4149	0.0249
22.40	22.3749	- 0.025	22.4249	0.0249

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
22.41	22.3849	- 0.025	22.4348	0.0248
22.42	22.395	- 0.0249	22.4448	0.0248
22.43	22.405	- 0.0248	22.4547	0.0247
22.44	22.415	- 0.0248	22.4646	0.0246
22.45	22.425	- 0.0247	22.4746	0.0246
22.46	22.435	- 0.0247	22.4845	0.0245
22.47	22.445	- 0.0246	22.4945	0.0245
22.48	22.455	- 0.0246	22.504	0.024
22.49	22.465	- 0.0245	22.514	0.024
22.50	22.476	- 0.024	22.524	0.024
22.51	22.486	- 0.024	22.534	0.024
22.52	22.496	- 0.024	22.544	0.024
22.53	22.506	- 0.024	22.554	0.024
22.54	22.516	- 0.024	22.564	0.024
22.55	22.526	- 0.024	22.574	0.024
22.56	22.536	- 0.024	22.584	0.024
22.57	22.546	- 0.024	22.594	0.024
22.58	22.556	- 0.024	22.604	0.024
22.59	22.566	- 0.024	22.614	0.024
22.60	22.576	- 0.024	22.624	0.024
22.61	22.586	- 0.024	22.634	0.024
22.62	22.596	- 0.024	22.644	0.024
22.63	22.606	- 0.024	22.654	0.024
22.64	22.616	- 0.024	22.664	0.024
22.65	22.626	- 0.024	22.674	0.024
22.66	22.636	- 0.024	22.683	0.023
22.67	22.646	- 0.024	22.693	0.023
22.68	22.657	- 0.023	22.703	0.023
22.69	22.667	- 0.023	22.713	0.023
22.70	22.677	- 0.023	22.723	0.023
22.71	22.687	- 0.023	22.733	0.023
22.72	22.697	- 0.023	22.743	0.023
22.73	22.707	- 0.023	22.753	0.023
22.74	22.717	- 0.023	22.763	0.023
22.75	22.727	- 0.023	22.773	0.023
22.76	22.737	- 0.023	22.783	0.023
22.77	22.747	- 0.023	22.793	0.023
22.78	22.757	- 0.023	22.803	0.023
22.79	22.767	- 0.023	22.813	0.023
22.80	22.777	- 0.023	22.823	0.023

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
22.81	22.787	- 0.023	22.833	0.023
22.82	22.797	- 0.023	22.843	0.023
22.83	22.807	- 0.023	22.853	0.023
22.84	22.817	- 0.023	22.863	0.023
22.85	22.827	- 0.023	22.872	0.022
22.86	22.837	- 0.023	22.882	0.022
22.87	22.848	- 0.022	22.892	0.022
22.88	22.858	- 0.022	22.902	0.022
22.89	22.868	- 0.022	22.912	0.022
22.90	22.878	- 0.022	22.922	0.022
22.91	22.888	- 0.022	22.932	0.022
22.92	22.898	- 0.022	22.942	0.022
22.93	22.908	- 0.022	22.952	0.022
22.94	22.918	- 0.022	22.962	0.022
22.95	22.928	- 0.022	22.972	0.022
22.96	22.938	- 0.022	22.982	0.022
22.97	22.948	- 0.022	22.992	0.022
22.98	22.958	- 0.022	23.002	0.022
22.99	22.968	- 0.022	23.012	0.022
23.00	22.978	- 0.022	23.022	0.022
23.01	22.988	- 0.022	23.032	0.022
23.02	22.998	- 0.022	23.042	0.022
23.03	23.008	- 0.022	23.052	0.022
23.04	23.018	- 0.022	23.062	0.022
23.05	23.028	- 0.022	23.071	0.021
23.06	23.038	- 0.022	23.081	0.021
23.07	23.049	- 0.021	23.091	0.021
23.08	23.059	- 0.021	23.101	0.021
23.09	23.069	- 0.021	23.111	0.021
23.10	23.079	- 0.021	23.121	0.021
23.11	23.089	- 0.021	23.131	0.021
23.12	23.099	- 0.021	23.141	0.021
23.13	23.109	- 0.021	23.151	0.021
23.14	23.119	- 0.021	23.161	0.021
23.15	23.129	- 0.021	23.171	0.021
23.16	23.139	- 0.021	23.181	0.021
23.17	23.149	- 0.021	23.191	0.021
23.18	23.159	- 0.021	23.201	0.021
23.19	23.169	- 0.021	23.211	0.021
23.20	23.179	- 0.021	23.221	0.021

Table 10. Tables of L(u)(continued).

u	L(u) + ½	L(-u)	L(u+½)	L(-u+½)
23.21	23.189	- 0.021	23.231	0.021
23.22	23.199	- 0.021	23.241	0.021
23.23	23.209	- 0.021	23.251	0.021
23.24	23.219	- 0.021	23.261	0.021
23.25	23.229	- 0.021	23.271	0.021
23.26	23.239	- 0.021	23.280	0.020
23.27	23.249	- 0.021	23.290	0.020
23.28	23.260	- 0.020	23.300	0.020
23.29	23.270	- 0.020	23.310	0.020
23.30	23.280	- 0.020	23.320	0.020
23.31	23.290	- 0.020	23.330	0.020
23.32	23.300	- 0.020	23.340	0.020
23.33	23.310	- 0.020	23.350	0.020
23.34	23.320	- 0.020	23.360	0.020
23.35	23.330	- 0.020	23.370	0.020
23.36	23.340	- 0.020	23.380	0.020
23.37	23.350	- 0.020	23.390	0.020
23.38	23.360	- 0.020	23.400	0.020
23.39	23.370	- 0.020	23.410	0.020
23.40	23.380	- 0.020	23.420	0.020
23.41	23.390	- 0.020	23.430	0.020
23.42	23.400	- 0.020	23.440	0.020
23.43	23.410	- 0.020	23.450	0.020
23.44	23.420	- 0.020	23.460	0.020
23.45	23.430	- 0.020	23.470	0.020
23.46	23.440	- 0.020	23.480	0.020
23.47	23.450	- 0.020	23.489	0.019
23.48	23.460	- 0.020	23.499	0.019
23.49	23.471	- 0.019	23.509	0.019
23.50	23.481	- 0.019	23.519	0.019
23.51	23.491	- 0.019	23.529	0.019
23.52	23.501	- 0.019	23.539	0.019
23.53	23.511	- 0.019	23.549	0.019
23.54	23.521	- 0.019	23.559	0.019
23.55	23.531	- 0.019	23.569	0.019
23.56	23.541	- 0.019	23.579	0.019
23.57	23.551	- 0.019	23.589	0.019
23.58	23.561	- 0.019	23.599	0.019
23.59	23.571	- 0.019	23.609	0.019
23.60	23.581	- 0.019	23.619	0.019

Table 10. Tables of L(u)(continued).

u	L(u) + Ɨ	L(-u)	L(u+Ɨ)	L(-u+Ɨ)
23.61	23.591	- 0.019	23.629	0.019
23.62	23.601	- 0.019	23.639	0.019
23.63	23.611	- 0.019	23.649	0.019
23.64	23.621	- 0.019	23.659	0.019
23.65	23.631	- 0.019	23.669	0.019
23.66	23.641	- 0.019	23.679	0.019
23.67	23.651	- 0.019	23.689	0.019
23.68	23.661	- 0.019	23.699	0.019
23.69	23.671	- 0.019	23.709	0.019
23.70	23.681	- 0.019	23.718	0.018
23.71	23.691	- 0.019	23.728	0.018
23.72	23.702	- 0.018	23.738	0.018
23.73	23.712	- 0.018	23.748	0.018
23.74	23.722	- 0.018	23.758	0.018
23.75	23.732	- 0.018	23.768	0.018
23.76	23.742	- 0.018	23.778	0.018
23.77	23.752	- 0.018	23.788	0.018
23.78	23.762	- 0.018	23.798	0.018
23.79	23.772	- 0.018	23.808	0.018
23.80	23.782	- 0.018	23.818	0.018
23.81	23.792	- 0.018	23.828	0.018
23.82	23.802	- 0.018	23.838	0.018
23.83	23.812	- 0.018	23.848	0.018
23.84	23.822	- 0.018	23.858	0.018
23.85	23.832	- 0.018	23.868	0.018
23.86	23.842	- 0.018	23.878	0.018
23.87	23.852	- 0.018	23.888	0.018
23.88	23.862	- 0.018	23.898	0.018
23.89	23.872	- 0.018	23.908	0.018
23.90	23.882	- 0.018	23.918	0.018
23.91	23.892	- 0.018	23.928	0.018
23.92	23.902	- 0.018	23.938	0.018
23.93	23.912	- 0.018	23.948	0.018
23.94	23.922	- 0.018	23.957	0.017
23.95	23.932	- 0.018	23.967	0.017
23.96	23.943	- 0.017	23.977	0.017
23.97	23.953	- 0.017	23.987	0.017
23.98	23.963	- 0.017	23.997	0.017
23.99	23.973	- 0.017	24.007	0.017
24.00	23.983	- 0.017	24.017	0.017

Table 10. Tables of L(u)(continued).

u	L(u) + Ī	L(-u)	L(u+Ī)	L(-u+Ī)
24.01	23.993	- 0.017	24.027	0.017
24.02	24.003	- 0.017	24.037	0.017
24.03	24.013	- 0.017	24.047	0.017
24.04	24.023	- 0.017	24.057	0.017
24.05	24.033	- 0.017	24.067	0.017
24.06	24.043	- 0.017	24.077	0.017
24.07	24.053	- 0.017	24.087	0.017
24.08	24.063	- 0.017	24.097	0.017
24.09	24.073	- 0.017	24.107	0.017
24.10	24.083	- 0.017	24.117	0.017
24.11	24.093	- 0.017	24.127	0.017
24.12	24.103	- 0.017	24.137	0.017
24.13	24.113	- 0.017	24.147	0.017
24.14	24.123	- 0.017	24.157	0.017
24.15	24.133	- 0.017	24.167	0.017
24.16	24.143	- 0.017	24.177	0.017
24.17	24.153	- 0.017	24.187	0.017
24.18	24.163	- 0.017	24.197	0.017
24.19	24.173	- 0.017	24.207	0.017
24.20	24.183	- 0.017	24.216	0.016
24.21	24.193	- 0.017	24.226	0.016
24.22	24.204	- 0.016	24.236	0.016
24.23	24.214	- 0.016	24.246	0.016
24.24	24.224	- 0.016	24.256	0.016
24.25	24.234	- 0.016	24.266	0.016
24.26	24.244	- 0.016	24.276	0.016
24.27	24.254	- 0.016	24.286	0.016
24.28	24.264	- 0.016	24.296	0.016
24.29	24.274	- 0.016	24.306	0.016
24.30	24.284	- 0.016	24.316	0.016
24.31	24.294	- 0.016	24.326	0.016
24.32	24.304	- 0.016	24.336	0.016
24.33	24.314	- 0.016	24.346	0.016
24.34	24.324	- 0.016	24.356	0.016
24.35	24.334	- 0.016	24.366	0.016
24.36	24.344	- 0.016	24.376	0.016
24.37	24.354	- 0.016	24.386	0.016
24.38	24.364	- 0.016	24.396	0.016
24.39	24.374	- 0.016	24.406	0.016
24.40	24.384	- 0.016	24.416	0.016

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
24.41	24.394	- 0.016	24.426	0.016
24.42	24.404	- 0.016	24.436	0.016
24.43	24.414	- 0.016	24.446	0.016
24.44	24.424	- 0.016	24.456	0.016
24.45	24.434	- 0.016	24.466	0.016
24.46	24.444	- 0.016	24.476	0.016
24.47	24.454	- 0.016	24.485	0.015
24.48	24.464	- 0.016	24.495	0.015
24.49	24.4745	- 0.015	24.505	0.015
24.50	24.4845	- 0.015	24.515	0.015
24.51	24.4945	- 0.015	24.525	0.015
24.52	24.5046	- 0.015	24.535	0.015
24.53	24.5146	- 0.015	24.545	0.015
24.54	24.5247	- 0.015	24.555	0.015
24.55	24.5347	- 0.015	24.565	0.015
24.56	24.5447	- 0.015	24.575	0.015
24.57	24.5548	- 0.015	24.585	0.015
24.58	24.5648	- 0.015	24.595	0.015
24.59	24.5748	- 0.015	24.605	0.015
24.60	24.5849	- 0.015	24.615	0.015
24.61	24.5949	- 0.015	24.6249	0.0149
24.62	24.6049	- 0.015	24.6349	0.0149
24.63	24.615	- 0.0149	24.6449	0.0149
24.64	24.625	- 0.0149	24.6548	0.0148
24.65	24.635	- 0.0149	24.6648	0.0148
24.66	24.645	- 0.0148	24.6748	0.0148
24.67	24.655	- 0.0148	24.6847	0.0147
24.68	24.665	- 0.0148	24.6947	0.0147
24.69	24.675	- 0.0147	24.7047	0.0147
24.70	24.685	- 0.0147	24.7146	0.0146
24.71	24.695	- 0.0147	24.7246	0.0146
24.72	24.705	- 0.0146	24.7346	0.0146
24.73	24.715	- 0.0146	24.7445	0 0145
24.74	24.725	- 0.0146	24.7545	0.0145
24.75	24.735	- 0.0145	24.7645	0.0145
24.76	24.745	- 0.0145	24.774	0.014
24.77	24.755	- 0.0145	24.784	0.014
24.78	24.766	- 0.014	24.794	0.014
24.79	24.776	- 0.014	24.804	0.014
24.80	24.786	- 0.014	24.814	0.014

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
24.81	24.796	- 0.014	24.824	0.014
24.82	24.806	- 0.014	24.834	0.014
24.83	24.816	- 0.014	24.844	0.014
24.84	24.826	- 0.014	24.854	0.014
24.85	24.836	- 0.014	24.864	0.014
24.86	24.846	- 0.014	24.874	0.014
24.87	24.856	- 0.014	24.884	0.014
24.88	24.866	- 0.014	24.894	0.014
24.89	24.876	- 0.014	24.904	0.014
24.90	24.886	- 0.014	24.914	0.014
24.91	24.896	- 0.014	24.924	0.014
24.92	24.906	- 0.014	24.934	0.014
24.93	24.916	- 0.014	24.944	0.014
24.94	24.926	- 0.014	24.954	0.014
24.95	24.936	- 0.014	24.964	0.014
24.96	24.946	- 0.014	24.974	0.014
24.97	24.956	- 0.014	24.984	0.014
24.98	24.966	- 0.014	24.994	0.014
24.99	24.976	- 0.014	25.004	0.014
25.00	24.986	- 0.014	25.014	0.014
25.01	24.996	- 0.014	25.024	0.014
25.02	25.006	- 0.014	25.034	0.014
25.03	25.016	- 0.014	25.044	0.014
25.04	25.026	- 0.014	25.054	0.014
25.05	25.036	- 0.014	25.064	0.014
25.06	25.046	- 0.014	25.074	0.014
25.07	25.056	- 0.014	25.083	0.013
25.08	25.066	- 0.014	25.093	0.013
25.09	25.077	- 0.013	25.103	0.013
25.10	25.087	- 0.013	25.113	0.013
25.11	25.097	- 0.013	25.123	0.013
25.12	25.107	- 0.013	25.133	0.013
25.13	25.117	- 0.013	25.143	0.013
25.14	25.127	- 0.013	25.153	0.013
25.15	25.137	- 0.013	25.163	0.013
25.16	25.147	- 0.013	25.173	0.013
25.17	25.157	- 0.013	25.183	0.013
25.18	25.167	- 0.013	25.193	0.013
25.19	25.177	- 0.013	25.203	0.013
25.20	25.187	- 0.013	25.213	0.012

Table 10. Tables of L(u)(continued).

u	L(u) + Ɨ	L(-u)	L(u+Ɨ)	L(-u+Ɨ)
25.21	25.197	- 0.013	25.223	0.013
25.22	25.207	- 0.013	25.233	0.013
25.23	25.217	- 0.013	25.243	0.013
25.24	25.227	- 0.013	25.253	0.013
25.25	25.237	- 0.013	25.263	0.013
25.26	25.247	- 0.013	25.273	0.013
25.27	25.257	- 0.013	25.283	0.013
25.28	25.267	- 0.013	25.293	0.013
25.29	25.277	- 0.013	25.303	0.013
25.30	25.287	- 0.013	25.313	0.013
25.31	25.297	- 0.013	25.323	0.013
25.32	25.307	- 0.013	25.333	0.013
25.33	25.317	- 0.013	25.343	0.013
25.34	25.327	- 0.013	25.353	0.013
25.35	25.337	- 0.013	25.363	0.013
25.36	25.347	- 0.013	25.373	0.013
25.37	25.357	- 0.013	25.383	0.013
25.38	25.367	- 0.013	25.393	0.013
25.39	25.377	- 0.013	25.403	0.013
25.40	25.387	- 0.013	25.413	0.013
25.41	25.397	- 0.013	25.422	0.012
25.42	25.408	- 0.012	25.432	0.012
25.43	25.418	- 0.012	25.442	0.012
25.44	25.428	- 0.012	25.452	0.012
25.45	25.438	- 0.012	25.462	0.012
25.46	25.448	- 0.012	25.472	0.012
25.47	25.458	- 0.012	25.482	0.012
25.48	25.468	- 0.012	25.492	0.012
25.49	25.478	- 0.012	25.502	0.012
25.50	25.488	- 0.012	25.512	0.012
25.51	25.498	- 0.012	25.522	0.012
25.52	25.508	- 0.012	25.532	0.012
25.53	25.518	- 0.012	25.542	0.012
25.54	25.528	- 0.012	25.552	0.012
25.55	25.538	- 0.012	25.562	0.012
25.56	25.548	- 0.012	25.572	0.012
25.57	25.558	- 0.012	25.582	0.012
25.58	25.568	- 0.012	25.592	0.012
25.59	25.578	- 0.012	25.602	0.012
25.60	25.588	- 0.012	25.612	0.012

Table 10. Tables of L(u)(continued).

u	L(u) + ½	L(-u)	L(u+½)	L(-u+½)
25.61	25.598	- 0.012	25.622	0.012
25.62	25.608	- 0.012	25.632	0.012
25.63	25.618	- 0.012	25.642	0.012
25.64	25.628	- 0.012	25.652	0.012
25.65	25.638	- 0.012	25.662	0.012
25.66	25.648	- 0.012	25.672	0.012
25.67	25.658	- 0.012	25.682	0.012
25.68	25.668	- 0.012	25.692	0.012
25.69	25.678	- 0.012	25.702	0.012
25.70	25.688	- 0.012	25.712	0.012
25.71	25.698	- 0.012	25.722	0.012
25.72	25.708	- 0.012	25.732	0.012
25.73	25.718	- 0.012	25.742	0.012
25.74	25.728	- 0.012	25.752	0.012
25.75	25.738	- 0.012	25.762	0.012
25.76	25.748	- 0.012	25.772	0.012
25.77	25.758	- 0.012	25.781	0.011
25.78	25.769	- 0.011	25.791	0.011
25.79	25.779	- 0.011	25.801	0.011
25.80	25.789	- 0.011	25.811	0.011
25.81	25.799	- 0.011	25.821	0.011
25.82	25.809	- 0.011	25.831	0.011
25.83	25.819	- 0.011	25.841	0.011
25.84	25.829	- 0.011	25.851	0.011
25.85	25.839	- 0.011	25.861	0.011
25.86	25.849	- 0.011	25.871	0.011
25.87	25.859	- 0.011	25.881	0.011
25.88	25.869	- 0.011	25.891	0.011
25.89	25.879	- 0.011	25.901	0.011
25.90	25.889	- 0.011	25.911	0.011
25.91	25.899	- 0.011	25.921	0.011
25.92	25.909	- 0.011	25.931	0.011
25.93	25.919	- 0.011	25.941	0.011
25.94	25.929	- 0.011	25.951	0.011
25.95	25.939	- 0.011	25.961	0.011
25.96	25.949	- 0.011	25.971	0.011
25.97	25.959	- 0.011	25.981	0.011
25.98	25.969	- 0.011	25.991	0.011
25.99	25.979	- 0.011	26.001	0.011
26.00	25.989	- 0.011	26.011	0.011

Table 10. Tables of L(u)(continued).

u	L(u) + ½	L(-u)	L(u+½)	L(-u+½)
26.01	25.999	- 0.011	26.021	0.011
26.02	26.009	- 0.011	26.031	0.011
26.03	26.019	- 0.011	26.041	0.011
26.04	26.029	- 0.011	26.051	0.011
26.05	26.039	- 0.011	26.061	0.011
26.06	26.049	- 0.011	26.071	0.011
26.07	26.059	- 0.011	26.081	0.011
26.08	26.069	- 0.011	26.091	0.011
26.09	26.079	- 0.011	26.101	0.011
26.10	26.089	- 0.011	26.111	0.011
26.11	26.099	- 0.011	26.121	0.011
26.12	26.109	- 0.011	26.131	0.011
26.13	26.119	- 0.011	26.141	0.011
26.14	26.129	- 0.011	26.151	0.011
26.15	26.139	- 0.011	26.161	0.011
26.16	26.149	- 0.011	26.171	0.011
26.17	26.159	- 0.011	26.180	0.010
26.18	26.170	- 0.010	26.190	0.010
26.19	26.180	- 0.010	26.200	0.010
26.20	26.190	- 0.010	26.210	0.010
26.21	26.200	- 0.010	26.220	0.010
26.22	26.210	- 0.010	26.230	0.010
26.23	26.220	- 0.010	26.240	0.010
26.24	26.230	- 0.010	26.250	0.010
26.25	26.240	- 0.010	26.260	0.010
26.26	26.250	- 0.010	26.270	0.010
26.27	26.260	- 0.010	26.280	0.010
26.28	26.270	- 0.010	26.290	0.010
26.29	26.280	- 0.010	26.300	0.010
26.30	26.290	- 0.010	26.310	0.010
26.31	26.300	- 0.010	26.320	0.010
26.32	26.310	- 0.010	26.330	0.010
26.33	26.320	- 0.010	26.340	0.010
26.34	26.330	- 0.010	26.350	0.010
26.35	26.340	- 0.010	26.360	0.010
26.36	26.350	- 0.010	26.370	0.010
26.37	26.360	- 0.010	26.380	0.010
26.38	26.370	- 0.010	26.390	0.010
26.39	26.380	- 0.010	26.400	0.010
26.40	26.390	- 0.010	26.410	0.010

Table 10. Tables of L(u)(continued).

u	L(u) + ½	L(-u)	L(u+½)	L(-u+½)
26.41	26.400	- 0.010	26.420	0.010
26.42	26.410	- 0.010	26.430	0.010
26.43	26.420	- 0.010	26.440	0.010
26.44	26.430	- 0.010	26.450	0.010
26.45	26.440	- 0.010	26.460	0.010
26.46	26.450	- 0.010	26.470	0.010
26.47	26.460	- 0.010	26.480	0.010
26.48	26.470	- 0.010	26.490	0.010
26.49	26.480	- 0.010	26.500	0.010
26.50	26.490	- 0.010	26.510	0.010
26.51	26.500	- 0.010	26.520	0.010
26.52	26.510	- 0.010	26.530	0.010
26.53	26.520	- 0.010	26.540	0.010
26.54	26.530	- 0.010	26.550	0.010
26.55	26.540	- 0.010	26.560	0.010
26.56	26.550	- 0.010	26.570	0.010
26.57	26.560	- 0.010	26.580	0.010
26.58	26.570	- 0.010	26.590	0.010
26.59	26.580	- 0.010	26.600	0.010
26.60	26.590	- 0.010	26.609	0.009
26.61	26.601	- 0.009	26.619	0.009
26.62	26.611	- 0.009	26.629	0.009
26.63	26.621	- 0.009	26.639	0.009
26.64	26.631	- 0.009	26.649	0.009
26.65	26.641	- 0.009	26.659	0.009
26.66	26.651	- 0.009	26.669	0.009
26.67	26.661	- 0.009	26.679	0.009
26.68	26.671	- 0.009	26.689	0.009
26.69	26.681	- 0.009	26.699	0.009
26.70	26.691	- 0.009	26.709	0.009
26.71	26.701	- 0.009	26.719	0.009
26.72	26.711	- 0.009	26.729	0.009
26.73	26.721	- 0.009	26.739	0.009
26.74	26.731	- 0.009	26.749	0.009
26.75	26.741	- 0.009	26.759	0.009
26.76	26.751	- 0.009	26.769	0.009
26.77	26.761	- 0.009	26.779	0.009
26.78	26.771	- 0.009	26.789	0.009
26.79	26.781	- 0.009	26.799	0.009
26.80	26.791	- 0.009	26.809	0.009

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
26.81	26.801	- 0.009	26.819	0.009
26.82	26.811	- 0.009	26.829	0.009
26.83	26.821	- 0.009	26.839	0.009
26.84	26.831	- 0.009	26.849	0.009
26.85	26.841	- 0.009	26.859	0.009
26.86	26.851	- 0.009	26.869	0.009
26.87	26.861	- 0.009	26.879	0.009
26.88	26.871	- 0.009	26.889	0.009
26.89	26.881	- 0.009	26.899	0.009
26.90	26.891	- 0.009	26.909	0.009
26.91	26.901	- 0.009	26.919	0.009
26.92	26.911	- 0.009	26.929	0.009
26.93	26.921	- 0.009	26.939	0.009
26.94	26.931	- 0.009	26.949	0.009
26.95	26.941	- 0.009	26.959	0.009
26.96	26.951	- 0.009	26.969	0.009
26.97	26.961	- 0.009	26.979	0.009
26.98	26.971	- 0.009	26.989	0.009
26.99	26.981	- 0.009	26.999	0.009
27.00	26.991	- 0.009	27.009	0.009
27.01	27.001	- 0.009	27.019	0.009
27.02	27.011	- 0.009	27.029	0.009
27.03	27.021	- 0.009	27.039	0.009
27.04	27.031	- 0.009	27.049	0.009
27.05	27.041	- 0.009	27.059	0.009
27.06	27.051	- 0.009	27.069	0.009
27.07	27.061	- 0.009	27.079	0.009
27.08	27.071	- 0.009	27.088	0.008
27.09	27.082	- 0.008	27.098	0.008
27.10	27.092	- 0.008	27.108	0.008
27.11	27.102	- 0.008	27.118	0.008
27.12	27.112	- 0.008	27.128	0.008
27.13	27.122	- 0.008	27.138	0.008
27.14	27.132	- 0.008	27.148	0.008
27.15	27.142	- 0.008	27.158	0.008
27.16	27.152	- 0.008	27.168	0.008
27.17	27.162	- 0.008	27.178	0.008
27.18	27.172	- 0.008	27.188	0.008
27.19	27.182	- 0.008	27.198	0.008
27.20	27.192	- 0.008	27.208	0.008

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
27.21	27.202	- 0.008	27.218	0.008
27.22	27.212	- 0.008	27.228	0.008
27.23	27.222	- 0.008	27.238	0.008
27.24	27.232	- 0.008	27.248	0.008
27.25	27.242	- 0.008	27.258	0.008
27.26	27.252	- 0.008	27.268	0.008
27.27	27.262	- 0.008	27.278	0.008
27.28	27.272	- 0.008	27.288	0.008
27.29	27.282	- 0.008	27.298	0.008
27.30	27.292	- 0.008	27.308	0.008
27.31	27.302	- 0.008	27.318	0.008
27.32	27.312	- 0.008	27.328	0.008
27.33	27.322	- 0.008	27.338	0.008
27.34	27.332	- 0.008	27.348	0.008
27.35	27.342	- 0.008	27.358	0.008
27.36	27.352	- 0.008	27.368	0.008
27.37	27.362	- 0.008	27.378	0.008
27.38	27.372	- 0.008	27.388	0.008
27.39	27.382	- 0.008	27.398	0.008
27.40	27.392	- 0.008	27.408	0.008
27.41	27.402	- 0.008	27.418	0.008
27.42	27.412	- 0.008	27.428	0.008
27.43	27.422	- 0.008	27.438	0.008
27.44	27.432	- 0.008	27.448	0.008
27.45	27.442	- 0.008	27.458	0.008
27.46	27.452	- 0.008	27.468	0.008
27.47	27.462	- 0.008	27.478	0.008
27.48	27.472	- 0.008	27.488	0.008
27.49	27.482	- 0.008	27.498	0.008
27.50	27.492	- 0.008	27.508	0.008
27.51	27.502	- 0.008	27.518	0.008
27.52	27.512	- 0.008	27.528	0.008
27.53	27.522	- 0.008	27.538	0.008
27.54	27.532	- 0.008	27.548	0.008
27.55	27.542	- 0.008	27.558	0.008
27.56	27.552	- 0.008	27.568	0.008
27.57	27.562	- 0.008	27.578	0.008
27.58	27.572	- 0.008	27.588	0.008
27.59	27.582	- 0.008	27.598	0.008
27.60	27.592	- 0.008	27.608	0.008

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
27.61	27.602	- 0.008	27.618	0.008
27.62	27.612	- 0.008	27.628	0.008
27.63	27.622	- 0.008	27.637	0.007
27.64	27.633	- 0.007	27.647	0.007
27.65	27.643	- 0.007	27.657	0.007
27.66	27.653	- 0.007	27.667	0.007
27.67	27.663	- 0.007	27.677	0.007
27.68	27.673	- 0.007	27.687	0.007
27.69	27.683	- 0.007	27.697	0.007
27.70	27.693	- 0.007	27.707	0.007
27.71	27.703	- 0.007	27.717	0.007
27.72	27.713	- 0.007	27.727	0.007
27.73	27.723	- 0.007	27.737	0.007
27.74	27.733	- 0.007	27.747	0.007
27.75	27.743	- 0.007	27.757	0.007
27.76	27.753	- 0.007	27.767	0.007
27.77	27.763	- 0.007	27.777	0.007
27.78	27.773	- 0.007	27.787	0.007
27.79	27.783	- 0.007	27.797	0.007
27.80	27.793	- 0.007	27.807	0.007
27.81	27.803	- 0.007	27.817	0.007
27.82	27.813	- 0.007	27.827	0.007
27.83	27.823	- 0.007	27.837	0.007
27.84	27.833	- 0.007	27.847	0.007
27.85	27.843	- 0.007	27.857	0.007
27.86	27.853	- 0.007	27.867	0.007
27.87	27.863	- 0.007	27.877	0.007
27.88	27.873	- 0.007	27.887	0.007
27.89	27.883	- 0.007	27.897	0.007
27.90	27.893	- 0.007	27.907	0.007
27.91	27.903	- 0.007	27.917	0.007
27.92	27.913	- 0.007	27.927	0.007
27.93	27.923	- 0.007	27.937	0.007
27.94	27.933	- 0.007	27.947	0.007
27.95	27.943	- 0.007	27.957	0.007
27.96	27.953	- 0.007	27.967	0.007
27.97	27.963	- 0.007	27.977	0.007
27.98	27.973	- 0.007	27.987	0.007
27.99	27.983	- 0.007	27.997	0.007
28.00	27.993	- 0.007	28.007	0.007

Table 10. Tables of L(u)(continued).

u	L(u) + Ī	L(-u)	L(u+Ī)	L(-u+Ī)
28.01	28.003	- 0.007	28.017	0.007
28.02	28.013	- 0.007	28.027	0.007
28.03	28.023	- 0.007	28.037	0.007
28.04	28.033	- 0.007	28.047	0.007
28.05	28.043	- 0.007	28.057	0.007
28.06	28.053	- 0.007	28.067	0.007
28.07	28.063	- 0.007	28.077	0.007
28.08	28.073	- 0.007	28.087	0.007
28.09	28.083	- 0.007	28.097	0.007
28.10	28.093	- 0.007	28.107	0.007
28.11	28.103	- 0.007	28.117	0.007
28.12	28.113	- 0.007	28.127	0.007
28.13	28.123	- 0.007	28.137	0.007
28.14	28.133	- 0.007	28.147	0.007
28.15	28.143	- 0.007	28.157	0.007
28.16	28.153	- 0.007	28.167	0.007
28.17	28.163	- 0.007	28.177	0.007
28.18	28.173	- 0.007	28.187	0.007
28.19	28.183	- 0.007	28.197	0.007
28.20	28.193	- 0.007	28.207	0.007
28.21	28.203	- 0.007	28.217	0.007
28.22	28.213	- 0.007	28.227	0.007
28.23	28.223	- 0.007	28.237	0.007
28.24	28.233	- 0.007	28.247	0.007
28.25	28.243	- 0.007	28.256	0.006
28.26	28.254	- 0.006	28.266	0.006
28.27	28.264	- 0.006	28.276	0.006
28.28	28.274	- 0.006	28.286	0.006
28.29	28.284	- 0.006	28.296	0.006
28.30	28.294	- 0.006	28.306	0.006
28.31	28.304	- 0.006	28.316	0.006
28.32	28.314	- 0.006	28.326	0.006
28.33	28.324	- 0.006	28.336	0.006
28.34	28.334	- 0.006	28.346	0.006
28.35	28.344	- 0.006	28.356	0.006
28.36	28.354	- 0.006	28.366	0.006
28.37	28.364	- 0.006	28.376	0.006
28.38	28.374	- 0.006	28.386	0.006
28.39	28.384	- 0.006	28.396	0.006
28.40	28.394	- 0.006	28.406	0.006

Table 10. Tables of L(u)(continued).

u	L(u) + Ī	L(-u)	L(u+Ī)	L(-u+Ī)
28.41	28.404	- 0.006	28.416	0.006
28.42	28.414	- 0.006	28.426	0.006
28.43	28.424	- 0.006	28.436	0.006
28.44	28.434	- 0.006	28.446	0.006
28.45	28.444	- 0.006	28.456	0.006
28.46	28.454	- 0.006	28.466	0.006
28.47	28.464	- 0.006	28.476	0.006
28.48	28.474	- 0.006	28.486	0.006
28.49	28.484	- 0.006	28.496	0.006
28.50	28.494	- 0.006	28.506	0.006
28.51	28.504	- 0.006	28.516	0.006
28.52	28.514	- 0.006	28.526	0.006
28.53	28.524	- 0.006	28.536	0.006
28.54	28.534	- 0.006	28.546	0.006
28.55	28.544	- 0.006	28.556	0.006
28.56	28.554	- 0.006	28.566	0.006
28.57	28.564	- 0.006	28.576	0.006
28.58	28.574	- 0.006	28.586	0.006
28.59	28.584	- 0.006	28.596	0.006
28.60	28.594	- 0.006	28.606	0.006
28.61	28.604	- 0.006	28.616	0.006
28.62	28.614	- 0.006	28.626	0.006
28.63	28.624	- 0.006	28.636	0.006
28.64	28.634	- 0.006	28.646	0.006
28.65	28.644	- 0.006	28.656	0.006
28.66	28.654	- 0.006	28.666	0.006
28.67	28.664	- 0.006	28.676	0.006
28.68	28.674	- 0.006	28.686	0.006
28.69	28.684	- 0.006	28.696	0.006
28.70	28.694	- 0.006	28.706	0.006
28.71	28.704	- 0.006	28.716	0.006
28.72	28.714	- 0.006	28.726	0.006
28.73	28.724	- 0.006	28.736	0.006
28.74	28.734	- 0.006	28.746	0.006
28.75	28.744	- 0.006	28.756	0.006
28.76	28.754	- 0.006	28.766	0.006
28.77	28.764	- 0.006	28.776	0.006
28.78	28.774	- 0.006	28.786	0.006
28.79	28.784	- 0.006	28.796	0.006
28.80	28.794	- 0.006	28.806	0.006

Table 10. Tables of L(u)(continued).

u	L(u) + I	L(-u)	L(u+I)	L(-u+I)
28.81	28.804	- 0.006	28.816	0.006
28.82	28.814	- 0.006	28.826	0.006
28.83	28.824	- 0.006	28.836	0.006
28.84	28.834	- 0.006	28.846	0.006
28.85	28.844	- 0.006	28.856	0.006
28.86	28.854	- 0.006	28.866	0.006
28.87	28.864	- 0.006	28.876	0.006
28.88	28.874	- 0.006	28.886	0.006
28.89	28.884	- 0.006	28.896	0.006
28.90	28.894	- 0.006	28.906	0.006
28.91	28.904	- 0.006	28.916	0.006
28.92	28.914	- 0.006	28.926	0.006
28.93	28.924	- 0.006	28.936	0.006
28.94	28.934	- 0.006	28.946	0.006
28.95	28.944	- 0.006	28.956	0.006
28.96	28.954	- 0.006	28.966	0.006
28.97	28.964	- 0.006	28.976	0.006
28.98	28.9745	- 0.005	28.985	0.005
28.99	28.9845	- 0.005	28.995	0.005
29.00	28.9945	- 0.005	29.005	0.005
29.01	29.0045	- 0.005	29.015	0.005
29.02	29.0145	- 0.005	29.025	0.005
29.03	29.0245	- 0.005	29.035	0.005
29.04	29.0345	- 0.005	29.045	0.005
29.05	29.0445	- 0.005	29.055	0.005
29.06	29.0546	- 0.005	29.065	0.005
29.07	29.0646	- 0.005	29.075	0.005
29.08	29.0746	- 0.005	29.085	0.005
29.09	29.0846	- 0.005	29.095	0.005
29.10	29.0946	- 0.005	29.105	0.005
29.11	29.1046	- 0.005	29.115	0.005
29.12	29.1146	- 0.005	29.125	0.005
29.13	29.1246	- 0.005	29.135	0.005
29.14	29.1347	- 0.005	29.145	0.005
29.15	29.1447	- 0.005	29.155	0.005
29.16	29.1547	- 0.005	29.165	0.005
29.17	29.1647	- 0.005	29.175	0.005
29.18	29.1747	- 0.005	29.185	0.005
29.19	29.1847	- 0.005	29.195	0.005
29.20	29.1947	- 0.005	29.205	0.005

Table 10. Tables of L(u)(continued).

u	L(u) + ½	L(-u)	L(u+½)	L(-u+½)
29.21	29.2047	- 0.005	29.215	0.005
29.22	29.2148	- 0.005	29.225	0.005
29.23	29.2248	- 0.005	29.235	0.005
29.24	29.2348	- 0.005	29.245	0.005
29.25	29.2448	- 0.005	29.255	0.005
29.26	29.2548	- 0.005	29.265	0.005
29.27	29.2648	- 0.005	29.275	0.005
29.28	29.2748	- 0.005	29.285	0.005
29.29	29.2848	- 0.005	29.295	0.005
29.30	29.2948	- 0.005	29.305	0.005
29.31	29.3049	- 0.005	29.315	0.005
29.32	29.3149	- 0.005	29.325	0.005
29.33	29.3249	- 0.005	29.335	0.005
29.34	29.3349	- 0.005	29.345	0.005
29.35	29.3449	- 0.005	29.355	0.005
29.36	29.3549	- 0.005	29.365	0.005
29.37	29.3649	- 0.005	29.375	0.005
29.38	29.3749	- 0.005	29.385	0.005
29.39	29.3849	- 0.005	29.3949	0.0049
29.40	29.395	- 0.0049	29.4049	0.0049
29.41	29.405	- 0.0049	29.4149	0.0049
29.42	29.415	- 0.0049	29.4249	0.0049
29.43	29.425	- 0.0049	29.4349	0.0049
29.44	29.435	- 0.0049	29.4449	0.0049
29.45	29.445	- 0.0049	29.4549	0.0049
29.46	29.455	- 0.0049	29.4649	0.0049
29.47	29.465	- 0.0049	29.4749	0.0049
29.48	29.475	- 0.0048	29.4848	0.0048
29.49	29.485	- 0.0048	29.4948	0.0048
29.50	29.495	- 0.0048	29.5048	0.0048
29.51	29.505	- 0.0048	29.5148	0.0048
29.52	29.515	- 0.0048	29.5248	0.0048
29.53	29.525	- 0.0048	29.5348	0.0048
29.54	29.535	- 0.0048	29.5448	0.0048
29.55	29.545	- 0.0048	29.5548	0.0048
29.56	29.555	- 0.0048	29.5648	0.0048
29.57	29.565	- 0.0047	29.5747	0.0047
29.58	29.575	- 0.0047	29.5847	0.0047
29.59	29.585	- 0.0047	29.5947	0.0047
29.60	29.595	- 0.0047	29.6047	0.0047

Table 10. Tables of L(u)(continued).

u	L(u) + ½	L(-u)	L(u+½)	L(-u+½)
29.61	29.605	- 0.0047	29.6147	0.0047
29.62	29.615	- 0.0047	29.6247	0.0047
29.63	29.625	- 0.0047	29.6347	0.0047
29.64	29.635	- 0.0047	29.6447	0.0047
29.65	29.645	- 0.0047	29.6547	0.0047
29.66	29.655	- 0.0046	29.6646	0.0046
29.67	29.665	- 0.0046	29.6746	0.0046
29.68	29.675	- 0.0046	29.6846	0.0046
29.69	29.685	- 0.0046	29.6946	0.0046
29.70	29.695	- 0.0046	29.7046	0.0046
29.71	29.705	- 0.0046	29.7146	0.0046
29.72	29.715	- 0.0046	29.7246	0.0046
29.73	29.725	- 0.0046	29.7346	0.0046
29.74	29.735	- 0.0046	29.7446	0.0046
29.75	29.745	- 0.0046	29.7545	0.0045
29.76	29.755	- 0.0045	29.7645	0.0045
29.77	29.765	- 0.0045	29.7745	0.0045
29.78	29.775	- 0.0045	29.7845	0.0045
29.79	29.785	- 0.0045	29.7945	0.0045
29.80	29.795	- 0.0045	29.8045	0.0045
29.81	29.805	- 0.0045	29.8145	0.0045
29.82	29.815	- 0.0045	29.8245	0.0045
29.83	29.825	- 0.0045	29.8345	0.0045
29.84	29.835	- 0.0045	29.8445	0.0045
29.85	29.846	- 0.004	29.854	0.004
29.86	29.856	- 0.004	29.864	0.004
29.87	29.866	- 0.004	29.874	0.004
29.88	29.876	- 0.004	29.884	0.004
29.89	29.886	- 0.004	29.894	0.004
29.90	29.896	- 0.004	29.904	0.004
29.91	29.906	- 0.004	29.914	0.004
29.92	29.916	- 0.004	29.924	0.004
29.93	29.926	- 0.004	29.934	0.004
29.94	29.936	- 0.004	29.944	0.004
29.95	29.946	- 0.004	29.954	0.004
29.96	29.956	- 0.004	29.964	0.004
29.97	29.966	- 0.004	29.974	0.004
29.98	29.976	- 0.004	29.984	0.004
29.99	29.986	- 0.004	29.994	0.004
30.00	29.996	- 0.004	30.004	0.004

APPENDIX D

TABLES OF COMPARISON SERIES

Comparison series.--In Chapter III it was pointed out that if all $A_k + H_k$ in a given expansion of H_o were equal to or smaller than the corresponding ${}_mA_k + {}_mH_k$ of another expansion, of which the approximants are termed a comparison series, then H_{on} is an equal or better approximation to H_o than is ${}_mH_{on}$. The tables in this appendix give the values,

$$D = {}_mH_{on} - H_o$$

for comparison series having a fixed value,

$${}_mA_k + {}_mH_k = M$$

at each stage. The comparison series selected for tabulation have values of M ranging from minus ten to plus six decibels, and D is calculated for the first seven approximants in each case.

Method of calculation.--Since ${}_mA_k + {}_mH_k = M$, having the same value for all k, ${}_ma_{km}G_k = m$, a constant such that M = 10 log m. By substituting m for r_k in equation (29) the following expansion for G_o is obtained:

$$G_{o\infty} = \cfrac{G_o}{m + \cfrac{1}{\cfrac{m}{1-m} + \cfrac{1}{m + \cfrac{1}{\cfrac{m}{1-m} + \cdots}}}}$$

Letting $\frac{1}{m^2} - \frac{1}{m} = \beta$, the above expansion may be rewritten in the form:

$$G_o = \cfrac{\frac{G_o}{m}}{1 + \cfrac{\beta}{1 + \cfrac{\beta}{1 + \cfrac{\beta}{1 + \cdots}}}}$$

The following expressions involving successive approximants are readily obtained.

$$m\left(\frac{m^{G_{oo}}}{G_o}\right) = \frac{1}{1}$$

$$m\left(\frac{m^{G_{o1}}}{G_o}\right) = \frac{1}{1 + \beta}$$

$$m\left(\frac{m^{G_{o2}}}{G_o}\right) = \frac{1 + \beta}{1 + 2\beta}$$

$$m\left(\frac{m^{G_{o3}}}{G_o}\right) = \frac{1 + 2\beta}{1 + 3\beta + \beta^2}$$

Let $m\left(\frac{m^{G_{on}}}{G_o}\right) = \frac{P_n(\beta)}{P_{n+1}(\beta)}$. The coefficients of ascending powers of β in the polynomials $P_n(\beta)$ are given by the following table.

Table 11. Coefficients of β^k in $P_n(\beta)$

k:	0	1	2	3	4
P_0	1	0	0	0	0
P_1	1	0	0	0	0
P_2	1	1	0	0	0
P_3	1	2	0	0	0
P_4	1	3	1	0	0
P_5	1	4	3	0	0
P_6	1	5	6	1	0
P_7	1	6	10	4	0
P_8	1	7	15	10	1
P_9	1	8	21	20	5

Each coefficient is the sum of the one directly above it and the one two spaces above and one to the left.

For each value of m the corresponding β is obtained, the polynomials $P_n(\beta)$ are evaluated, and the values of $m(\frac{m^G on}{G_o})$ are calculated. The appropriate logarithmic values for the table are then readily computed.

<u>Tables</u>.--Table 12 gives in decibels the difference, D, between the nth approximant of the comparison series and the prescribed function.

Table 12. Values of D for Comparison Series

M n=	0	1	2	3	4	5	6
-10.0	10.00	-9.59	7.01	-6.65	5.31	-4.99	4.14
- 6.0	6.00	-5.10	3.16	-2.56	1.76	-1.38	0.99
- 4.0	4.00	-2.81	1.47	-0.94	0.54	-0.33	0.20
- 3.0	3.00	-1.75	0.79	-0.41	0.20	-0.10	0.05
- 2.0	2.00	-0.85	0.29	-0.11	0.04	-0.01	0.01
- 1.0	1.00	-0.23	0.05	-0.01	0.00	0.00	0.00
- 0.5	0.50	-0.06	0.01	0.00	0.00	0.00	0.00
0.5	-0.05	-0.06	-0.01	0.00	0.00	0.00	0.00
1.0	-1.00	-0.23	-0.06	-0.01	0.00	0.00	0.00
2.0	-2.00	-0.85	-0.43	-0.23	-0.13	-0.07	-0.04
3.0	-3.00	-1.75	-1.23	-0.96	-0.78	-0.66	-0.60
4.0	-4.00	-2.81	-2.36	-2.13	-2.00	-1.93	-1.88
6.0	-6.00	-5.10	-4.86	-4.78	-4.76	-4.75	-4.74

Curves of the values of D are plotted graphically in Figure 58.

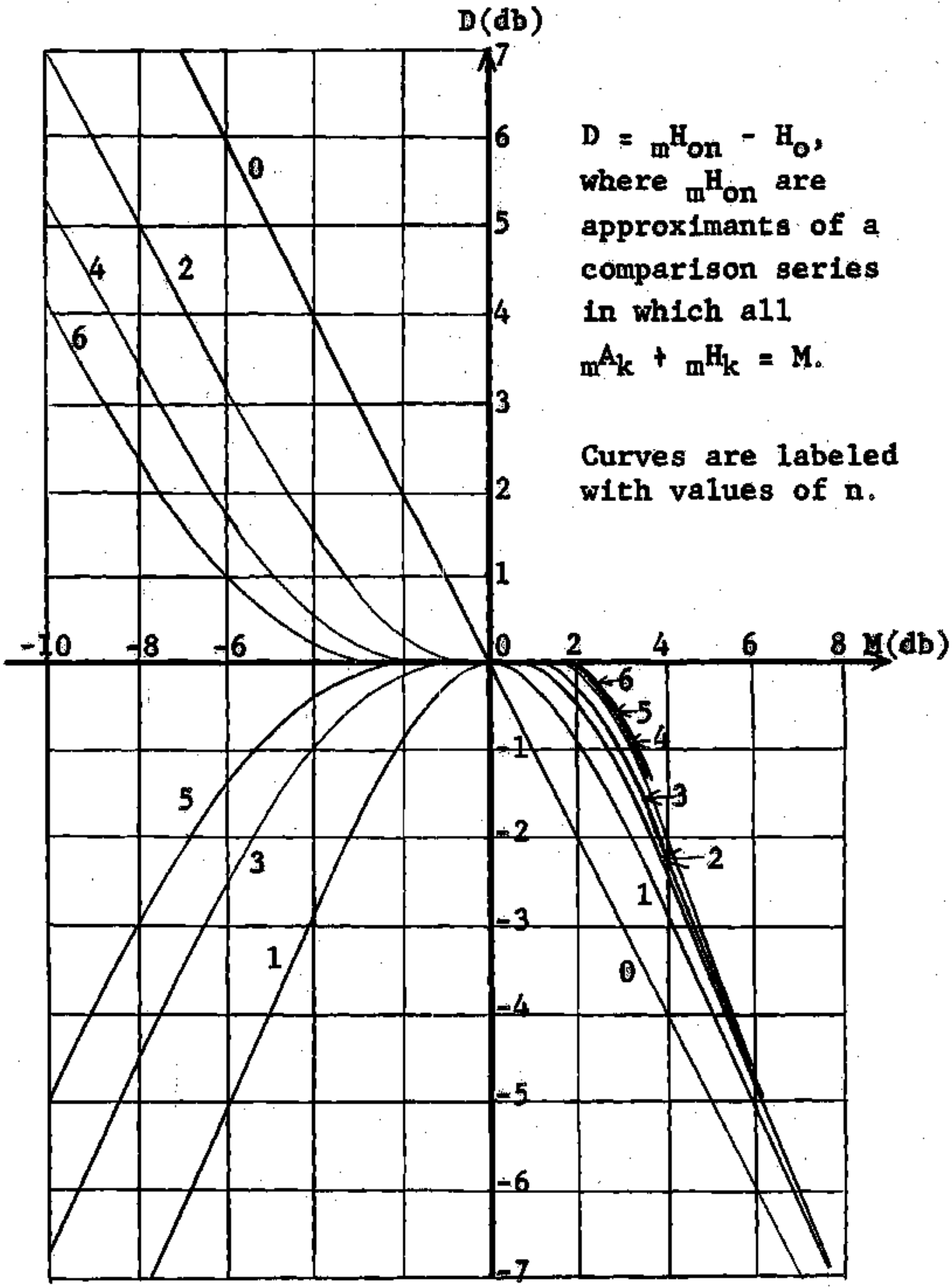

Fig. 58. Values of D for Comparison Series.

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19. _Ibid._, p. 403.

20. Wall, _op. cit._, p. 29.

21. Stieltjes, _op. cit._, p. 403.

22. _Ibid._, p. 467

23. Wall, _op. cit._, pp. 45-47.

24. _Ibid._, pp. 36-37.

25. _Ibid._, p. 15.

VITA

Elliott Carr Cutler, Jr. was born in Boston, Massachusetts, on June 15, 1920. He attended the Noble and Greenough School in Dedham, Massachusetts, and graduated from the United States Military Academy at West Point with the Bachelor of Science degree in 1942.

He was commissioned in the infantry, and served in Europe with the 76th Division in the Second World War. He has attended the Infantry School at Ft. Benning, Georgia, and the Armored School at Ft. Knox, Kentucky, and is a graduate of the Command and General Staff College at Ft. Leavenworth, Kansas. In 1950 and 1951 he served as operations officer of the 19th Infantry, 24th Division, in the Korean War. He holds the Legion of Merit, Bronze Star Medal, Purple Heart, and Combat Infantry Badge with star.

From 1945 to 1948 he was an instructor in the Department of Electricity at West Point. After receiving the degree of Master of Science in Electrical Engineering from the Georgia Institute of Technology at Atlanta, Georgia, in 1954, he returned to West Point as Associate Professor. In 1958 he was appointed Professor of Electrical Engineering at the United States Military Academy.